Čech Cohomology

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1 Sheaves

We will start a discussion on generalities regarding presheaves and sheaves. For more details, one can look at the first section of chapter 2 of [Har77].
Sheaves are objects which carry local data on a topological space. What this means precisely will become clear from the definitions and examples. To define sheaves, first we need to define a presheaf.

Definition. *For a topological space X, set $Top(X)$ to be the category whose objects are open subsets of X and morphisms are inclusions of open sets. Then a* ***presheaf*** *on X (with values in a category $\mathfrak{C}$) is just a contravariant functor from $Top(X) \to \mathfrak{C}$.*
More explicitly, a presheaf $\mathcal{F}$ on X consists of the data of an object $\mathcal{F}(U)$ of $\mathfrak{C}$ associated to every open set U of X along with "restriction" morphisms $r_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ corresponding to inclusions $V \hookrightarrow U$ of open sets of X such that if we have a chain of inclusions $W \hookrightarrow V \hookrightarrow U$, then $r_{UW} = r_{VW} \circ r_{UV}$.

Examples :

1. Fix an object A in a category $\mathfrak{C}$. Then $\mathcal{C}_A(U) = A$, with all the restriction morphisms being the identity morphism on A forms a presheaf on X called the **constant presheaf** with value A.

2. If X is a topological space (manifold, complex manifold, algebraic variety), then $\mathcal{O}_X(U) =$ space of real-valued continous functions (smooth functions, holomorphic functions, regular functions) on the open set U with the restriction maps $r_{UV}(f) = f\big|_V$ forms a presheaf on X with values in $\mathbb{R}$-algebras ($\mathbb{R}$-algebras, $\mathbb{C}$-algebras, k-algebras). We will see later that this is infact a sheaf, called the sheaf of real-valued continuous (real-valued smooth, holomorphic, regular) functions on X.

3. Similar to the last example, one can consider $\mathcal{F}(U) =$ the real-vector space of real-valued bounded functions on U, with the same restriction maps as the last example. This also forms a presheaf, which we will called the presheaf of bounded continuous functions on X.

4. Let $\mathfrak{C} = \mathfrak{Mod}(A)$, the category of modules over the ring A, then for an A-module M and a point $P \in X$, we can define the presheaf M_P by $M_P(U) = M$ if $P \in M$ and $M_P(U) = 0$ otherwise, with the restriction map being identity on M if the smaller open set contains P and the zero map otherwise. This is also a sheaf, known as the skyscraper sheaf at P with the value M.

5. Lastly, consider any continuous map $p : E \to X$, then we have a presheaf given by $\mathcal{F}(U) = \{s : U \to E \text{ continuous } \mid p \circ s = id_U\}$ the set of continuous sections of p over U, where the restriction maps again simple restrict the domain of the section. This will also turn out to be a sheaf, called the sheaf of sections of p.

We want to make the collection of presheaves into a category, i.e. we need a notion of morphisms between presheaves.

Definition. *A* ***morphism*** *$\phi : \mathcal{F} \to \mathcal{G}$ between two presheaves with values in $\mathfrak{C}$ is just a natural transformation between the functors $\mathcal{F} : Top(X) \to \mathfrak{C}$ and $\mathcal{G} : Top(X) \to \mathfrak{C}$.*
Again, more explicity ϕ consists of morphisms $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ in $\mathfrak{C}$ for every open set U of X, such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \Big\downarrow{\scriptstyle r_{UV}} & & \Big\downarrow{\scriptstyle s_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\phi(V)} & \mathcal{G}(V) \end{array}$$

where r_{UV} and s_{UV} denote restriction maps for $\mathcal{F}$ and $\mathcal{G}$ respectively.
This makes the collection of presheaves on X with values in $\mathfrak{C}$ into a category, denoted by $PSh_{\mathfrak{C}}(X)$.

We will only be working with the cases where $\mathfrak{C} = \mathfrak{Ab}$ category of abelian groups (called presheaves of abelian groups) or more generally $\mathfrak{C} = \mathfrak{Mod}(A)$ the category of modules over a commutative ring with unity, A (called presheaves of A-modules), and hence from now we will just write $PSh(X)$ where it is understood that we are talking about the category of presheaves on X with values in either of these categories, unless otherwise stated.
In these cases, given a presheaf $\mathcal{F}$, the object $\mathcal{F}(U)$ is atleast a set and hence contains elements, which, following the last example above, will be referred to as **sections** over U, and sections over all of X will sometimes be referred to as **global sections**. Moreover, for a section $s \in \mathcal{F}(U)$, we will denote its image under the restriction map r_{UV} of $\mathcal{F}$ by $s\big|_V$.

By a sub-presheaf of a presheaf $\mathcal{F}$ we mean a presheaf $\mathcal{G}$ such that for all open sets U, the modules $\mathcal{G}(U)$ are submodules of $\mathcal{F}(U)$ and the restriction maps of $\mathcal{G}$ are simply the restriction maps of $\mathcal{F}$ restricted to these submodules. Given a sub-presheaf $\mathcal{G} \hookrightarrow \mathcal{F}$ of $\mathcal{F}$, we can construct the quotient presheaf $\mathcal{F}/\mathcal{G}$ as $\mathcal{F}/\mathcal{G}(U) = \mathcal{F}(U)/\mathcal{G}(U)$ with the restriction maps being the ones induced by the restriction maps of $\mathcal{F}$.
Given a morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$, we can define the kernel presheaf $(\ker \phi)(U) = \ker(\phi(U))$, the image presheaf $(\operatorname{im}_{PSh} \phi)(U) = \operatorname{im}(\phi(U))$ as sub-presheaves of $\mathcal{F}$ and $\mathcal{G}$ respectively and $\operatorname{coker}_{PSh} \phi$ as the quotient presheaf $\mathcal{G}/\operatorname{im} \phi$.

Consider the category $PSh(X)$ of presheaves on a space X with values in $\mathfrak{Mod}(R)$

- It contains a zero object, namely the constant presheaf with the value 0, i.e. the trivial module.
- It contains direct sums and products, which coincide, given by $(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$ and the obvious restriction maps given by sums of restriction maps of the presheaves $\mathcal{F}$ and $\mathcal{G}$.
- The collection of morphisms $Hom(\mathcal{F}, \mathcal{G})$ forms an abelian group, where the sum of two morphisms is given by adding the morphisms over each open set.
- The kernel and cokernel presheaves as defined above, satisfy the expected universal properties for kernels and cokernels.

Together all of this shows that $PSh(X)$ forms an abelian category. Therefore we have a notion of exactness, and it is easy to see that a sequence of maps $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact at $\mathcal{G}$ iff for all open sets U, the corresponding sequences of modules $\mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$ are exact, since kernels, cokernels and images are defined by taking kernels, cokernels and images over each open set.
Finally, we will give the definition of a sheaf.

Definition. *A presheaf $\mathcal{F}$ is called a **sheaf**, given any open set U and an open cover $\{U_i\}_{i \in I}$ of U, it satisfies the following two conditions:*

- *(Identity axiom) For two sections $s, t \in \mathcal{F}(U)$, if $s\big|_{U_i} = t\big|_{U_i}$ for all $i \in I$, then $s = t$.*
- *(Glueability axiom) For a collection of sections $\{s_i\}_{i \in I}$, where $s_i \in \mathcal{F}(U_i)$ such that $s_i\big|_{U_i \cap U_j} = s_j\big|_{U_i \cap U_j}$, there exists a section $s \in \mathcal{F}(U)$ such that $s\big|_{U_i} = s_i$.*

A morphism of sheaves from $\mathcal{F}$ to $\mathcal{G}$ is simply a morphism of presheaves $\mathcal{F} \to \mathcal{G}$. We denote the category of sheaves by $Sh(X)$

It is an easy exercise to check that the examples 2,4 and 5 of presheaves given above are actually sheaves. For 2 and 5, it follows from the fact that continuous (smooth, holomorphic, regular) functions on open subsets can be glued together if they agree on the intersections, which is exactly the glueability axiom above, and the fact that these functions are determined by their values at each point, which is infact a little stronger than the identity axiom. The example 3 turns out to not be a sheaf in general since, for example, if we take $X = \mathbb{R}$ the real line, then the restriction of the identity map to open intervals $]n, n+2[$ are bounded continuous functions but they do not glue together to give a bounded continuous function on the entire real line. This reflects the fact that boundedness is not a "local" condition. Similarly, in example 1, if we take the category $\mathfrak{C}$ to be the category of sets or modules over a ring, then the sections of the constant presheaf globally assign an element of A to the entire space. To see that this isn't a sheaf, simply take X to be the two point space with the discrete topology and consider the

2-element cover consisting of each of the points.

It is easy to see the that given a morphism of sheaves, the kernel presheaf is actually a sheaf. But the same is not true for the image presheaf. Consider the example,

$$\begin{aligned} \mathcal{O}_{\mathbb{CP}^1} &\to \mathbb{C}_P \oplus \mathbb{C}_Q \\ f &\mapsto (f(P), f(Q)) \end{aligned}$$

Where $\mathcal{O}_{\mathbb{CP}^1}$ is the sheaf of holomorphic functions on the Riemann sphere and $\mathbb{C}_P \oplus \mathbb{C}_Q$ is the direct sum of the skyscraper sheaves at P and Q with value $\mathbb{C}$. Then it is easy to see, that over the open sets $\mathbb{CP}^1 - P$ and $\mathbb{CP}^1 - Q$, the map of sections is surjective. But since there are no nonconstant holomorphic functions on all of $\mathbb{CP}^1$, the map on global sections is the diagonal embedding $\mathbb{C} \to \mathbb{C} \oplus \mathbb{C}$. But in the image presheaf, we have all sections of $\mathbb{C}_P \oplus \mathbb{C}_Q$ over $\mathbb{CP}^1 - P$ and $\mathbb{CP}^1 - Q$. $\mathbb{C}_P \oplus \mathbb{C}_Q$ has no non-zero sections over the intersection $\mathbb{CP}^1 - \{P, Q\}$, so every pair of sections over $\mathbb{CP}^1 - P$ and $\mathbb{CP}^1 - Q$ agree on the overlap. But then glueability would imply that all of $\mathbb{C} \oplus \mathbb{C}$ lies in the global sections of the image presheaf, which is not true (only the diagonal elements lie in the global sections of the image presheaf).
Generally this occurs because if a collection of sections in the image presheaf agree on intersections, it doesn't necessarily mean that their preimages would agree on the intersections, not allowing to glue in the domain sheaf to get a glued image. To get around this, we define the image sheaf to be the **sheafification** of the image presheaf. One can define the sheafification of an arbitrary presheaf, but here we only need to worry about the case of the image presheaf.

Definition. *Given a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$, we define the **image sheaf** $\operatorname{im}\phi$ to be the subsheaf of $\mathcal{G}$ given by,*

$$(\operatorname{im}\phi)(U) = \{s \in \mathcal{G}(U) \mid \exists \text{ a cover } \{U_i\}_{i\in I} \text{ of } U \text{ such that } s\big|_{U_i} \in \operatorname{im}(\phi(U_i))\}$$

With these definition of kernel and image sheafs, the category of sheaves also turns out to be an abelian category, but exactness can no longer be checked over each open set since the image sheaf is larger than the image presheaf.

Definition. *The stalk of a presheaf $\mathcal{F}$ at a point $P \in X$ is defined by the direct limit:*

$$\mathcal{F}_P = \varinjlim_{P \in U} \mathcal{F}(U)$$

The image of a section $s \in \mathcal{F}(U)$ over an open set U containing P in the stalk at P is denoted by s_P.

The stalk at P can be interpreted as germs of sections of $\mathcal{F}$ at the point P, i.e. the elements of the stalk can be interpreted as equivalence classes of pairs (U, f), where U is an open set containing U and $f \in \mathcal{F}(U)$, such that $(U, f) \sim (V, g)$ if there exists an open set $W \subset U \cap V$ containing P such that $f\big|_W = g\big|_W$. In particular, in the second example above, the stalk at a point is in fact, the collection of germs of continuous (smooth, holomorphic, regular) functions at that point. Note that if the presheaf $\mathcal{F}$ takes values in $\mathfrak{Mod}(R)$, then the stalks also inherit the structure of an R-module.
Given a morphism $\phi : \mathcal{F} \to \mathcal{G}$ of presheaves, it induces a morphism $\phi_P : \mathcal{F}_P \to \mathcal{G}_P$ on the stalks at all the points $P \in X$. Since taking direct limits in the category of modules is exact, given an exact sequence $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$, the induced sequences of the stalks at all points $P \in X$, $\mathcal{F}_P \to \mathcal{G}_P \to \mathcal{H}_P$ will also be exact. The converse however is not true as again in the example on the Riemann Sphere above, the maps on the stalks are all surjective, but the maps on global sections is clearly not. However the converse does hold for sheaves, and hence exactness for sheaves is the same as exactness for stalks at all points. We won't be using this so we omit the proof. Lastly, we will define the global sections functor.

Definition. *Given a topological space X, the functor $\Gamma(X, -) : Sh(X) \to \mathfrak{Ab}$ (or $\mathfrak{Mod}(A)$) defined by*

$$\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$$

*is called the **global sections functor**. Given a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$, this functor maps it to the induced map on global sections $\Gamma(X, \phi) = \phi(X)$.*

From our discussion above it is easy to see that this functor is left exact. Čech cohomology is usually used to compute the right derived functors of the global sections functor. In this project we will independently study Čech cohomology and show that the de Rham cohomology of a manifold is infact the Čech cohomology of an extremely simple sheaf.

2 Čech Cohomology

In this section we will be using some terms and elementary results from homological algebra. For reference, one can look at the first chapter of [Wei94]. Our exposition follows chapter 3 section 4 of [Har77].
We will now define the Čech Cohomology groups of a sheaf $\mathcal{F}$ with values in $\mathfrak{Ab}$, with respect to a cover of the space X. Let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open covering of X, where I is a totally ordered indexing set. For a sequence of indices $i_0 < \cdots < i_p$, set $U_{i_0\ldots i_p} = \cap_{k=0}^{p} U_{i_k}$. For $p \geq 0$, define the the module of p-cochains to be

$$C^p(\mathfrak{U}, \mathcal{F}) = \prod_{i_0<\cdots<i_p} \mathcal{F}(U_{i_0\ldots i_p})$$

An element $\alpha \in C^p(\mathfrak{U}, \mathcal{F})$ consists of elements $\alpha_{i_0\ldots i_p} \in \mathcal{F}(U_{i_0\ldots i_p})$ for all increasing sets of indices $i_0 < \cdots < i_p$. We extend this to all possible $p+1$ tuples of indices $i_0, \ldots, i_p$ by setting $\alpha_{i_0\ldots i_p} = 0$ if any two of the indices $i_0, \ldots, i_p$ are equal and for any arbitrary collection of distinct indices $i_0, \ldots, i_p$, we set $\alpha_{i_0\ldots i_p} = \mathrm{sgn}(\sigma)\alpha_{\sigma(i_0)\ldots\sigma(i_p)}$ where σ is the unique permutation of the indices $i_0 \ldots i_p$ such that $\sigma(i_0) < \cdots < \sigma(i_p)$ and $\mathrm{sgn}(\sigma)$ is the sign of the permutation σ. This more generally sets $\alpha_{i_0\ldots i_p} = \mathrm{sgn}(\sigma)\alpha_{\sigma(i_0)\ldots\sigma(i_p)}$ for all permutations σ since the sign map is a group homomorphism. We define the co-boundary map $d_p : C^p(\mathfrak{U}, \mathcal{F}) \to C^{p+1}(\mathfrak{U}, \mathcal{F})$ as,

$$(d_p\alpha)_{i_0\ldots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0\ldots\widehat{i_k}\ldots i_{p+1}}|_{U_{i_0\ldots i_{p+1}}}$$

where $\widehat{i_k}$ means we omit i_k from the series of indices $i_0 \ldots i_{p+1}$. Since it is clear what open set we are restricting to in the RHS, we will omit the restriction in future computations. We initially make this definition only for increasing sequences of indices $i_0 < \cdots < i_{p+1}$, but if we consider the sum in the RHS for arbitrary indices $i_0, \ldots, i_{p+1}$ and swap i_r and i_{r+1}, (i.e. consider indices $j_0, \ldots j_p$, where $j_k = i_k$ for $k \neq r, r+1$ and $j_r = i_{r+1}$, $j_{r+1} = i_r$) we get,

$$\begin{aligned}
\sum_{k=0}^{p+1}(-1)^k\alpha_{j_0\ldots\widehat{j_k}\ldots j_{p+1}} &= \sum_{k=0}^{r-1}(-1)^k\alpha_{j_0\ldots\widehat{j_k}\ldots j_{p+1}} + (-1)^r\alpha_{j_0\ldots\widehat{j_r}\ldots j_{p+1}} + (-1)^{r+1}\alpha_{j_0\ldots\widehat{j_{k+1}}\ldots j_{r+1}} + \sum_{k=r+1}^{p+1}(-1)^k\alpha_{j_0\ldots\widehat{j_k}\ldots j_{p+1}} \\
&= \sum_{k=0}^{r-1}(-1)^{k+1}\alpha_{i_0\ldots\widehat{i_k}\ldots i_{p+1}} + (-1)^r\alpha_{i_0\ldots\widehat{i_{r+1}}\ldots i_{p+1}} + (-1)^{r+1}\alpha_{i_0\ldots\widehat{i_k}\ldots j_{r+1}} + \sum_{k=r+1}^{p+1}(-1)^{k+1}\alpha_{i_0\ldots\widehat{i_k}\ldots i_{p+1}} \\
&= -\sum_{k=0}^{p+1}(-1)^k\alpha_{i_0\ldots\widehat{i_k}\ldots i_{p+1}}
\end{aligned}$$

Since adjacent transpositions generate all permutations, have sign (-1) and sign is a group homomorphism, this shows,

$$\sum_{k=0}^{p+1}(-1)^k\alpha_{\sigma(i_0)\ldots\widehat{\sigma(j_k)}\ldots\sigma i_{p+1}} = \mathrm{sgn}(\sigma)\sum_{k=0}^{p+1}(-1)^k\alpha_{i_0\ldots\widehat{i_k}\ldots i_{p+1}}$$

for all indices $i_0, \ldots, i_{p+1}$. This shows that our definition of the coboundary map above respects our extension of definition of $(d\alpha)_{i_0\ldots i_{p+1}}$ for all possible indices $i_0, \ldots, i_{p+1}$. For this to make a complex we need $d^{p+1} \circ d^p = 0$, but that can be easily seen, since,

$$
\begin{aligned}
(d^{p+1}(d^p\alpha))_{i_0\dots i_{p+2}} &= \sum_{k=0}^{p+2}(-1)^k(d^p\alpha)_{i_0\dots\widehat{i_k}\dots i_{p+2}} \\
&= \sum_{k=0}^{p+2}(-1)^k\left(\sum_{l=0}^{k-1}(-1)^l\alpha_{i_0\dots\widehat{i_l}\dots\widehat{i_k}\dots i_{p+2}} + \sum_{l=k+1}^{p+2}(-1)^{l-1}\alpha_{i_0\dots\widehat{i_k}\dots\widehat{i_l}\dots i_{p+2}}\right) \\
&= \sum_{k=0}^{p+2}\sum_{l=0}^{k-1}(-1)^{l+k}\alpha_{i_0\dots\widehat{i_l}\dots\widehat{i_k}\dots i_{p+2}} + \sum_{k=0}^{p+2}\sum_{l=k+1}^{p+2}(-1)^{l+k-1}\alpha_{i_0\dots\widehat{i_k}\dots\widehat{i_l}\dots i_{p+2}} \\
&= \sum_{k=0}^{p+2}\sum_{l=0}^{k-1}(-1)^{l+k}\alpha_{i_0\dots\widehat{i_l}\dots\widehat{i_k}\dots i_{p+2}} + \sum_{l=0}^{p+2}\sum_{k=l+1}^{p+2}(-1)^{l+k-1}\alpha_{i_0\dots\widehat{i_l}\dots\widehat{i_k}\dots i_{p+2}} \quad \text{(replace } l \text{ and } k\text{)} \\
&= \sum_{k=0}^{p+2}\sum_{l=0}^{k-1}(-1)^{l+k}\alpha_{i_0\dots\widehat{i_l}\dots\widehat{i_k}\dots i_{p+2}} + \sum_{k=0}^{p+2}\sum_{l=0}^{k-1}(-1)^{l+k-1}\alpha_{i_0\dots\widehat{i_l}\dots\widehat{i_k}\dots i_{p+2}} \quad \text{(interchanging the order of sum)} \\
&= 0
\end{aligned}
$$

Therefore, if we set $C^p(\mathfrak{U},\mathcal{F}) = 0$ for $p < 0$, we have a cochain complex $C^\bullet(\mathfrak{U},\mathcal{F})$ of abelian groups given by,

$$\dots \longrightarrow 0 \longrightarrow C^0(\mathfrak{U},\mathcal{F}) \xrightarrow{d^0} C^1(\mathfrak{U},\mathcal{F}) \xrightarrow{d^1} C^2(\mathfrak{U},\mathcal{F}) \xrightarrow{d^2} \dots$$

Definition. *The complex $C^\bullet(\mathfrak{U},\mathcal{F})$ is called the* ***Čech complex*** *of $\mathcal{F}$ with respect to the cover $\mathfrak{U}$. Its cohomology is called the* ***Čech cohomology*** *of $\mathcal{F}$ with respect to the cover $\mathfrak{U}$ and is denoted by $\check{H}^\star(\mathfrak{U},\mathcal{F})$, i.e.,*

$$\check{H}^p(\mathfrak{U},\mathcal{F}) = h^p(C^\bullet(\mathfrak{U},\mathcal{F}))$$

The 0th Čech cohomology group can be easily computed:

Lemma 2.1. $\check{H}^0(\mathfrak{U},\mathcal{F}) \cong \Gamma(X,\mathcal{F})$

Proof. Since $C^{-1}(\mathfrak{U},\mathcal{F}) = 0$, $\check{H}(\mathfrak{U},\mathcal{F}) = \ker d^0$. For $\alpha \in C^0(\mathfrak{U},\mathcal{F})$, $(d\alpha)_{ij} = \alpha_j - \alpha_i$. Consider the map $s \in \Gamma(X,\mathcal{F}) \mapsto (s\big|_{U_i})_{i\in I} \in C^0(\mathfrak{U},\mathcal{F})$. Since $(d(s\big|_{U_i})) = s\big|_{U_i\cap U_j} - s\big|_{U_i\cap U_j} = 0$, the image of this map lies in $\ker d^1 = \check{H}(\mathfrak{U},\mathcal{F})$. For any $\alpha \in \ker d^1$, $\alpha_i \in \mathcal{F}(U_i)$, with $(d\alpha)_{ij} = \alpha_j\big|_{U_i\cap U_j} - \alpha_i\big|_{U_i\cap U_j} = 0$, i.e. $\alpha_i\big|_{U_i\cap U_j} = \alpha_j\big|_{U_i\cap U_j}$. Now the injectivity and the surjectivity of this map follows from (and is equivalent to) the identity and glueability axioms of sheaves. ☐

Definition. *We define the* ***augmented Čech complex*** *of $\mathcal{F}$ w.r.t $\mathfrak{U}$, to be the complex,*

$$0 \longrightarrow \Gamma(X,\mathcal{F}) \longrightarrow C^0(\mathfrak{U},\mathcal{F}) \longrightarrow C^1(\mathfrak{U},\mathcal{F}) \longrightarrow \dots$$

where the second map is the one defined in the lemma 2.1. From the same lemma, it is clear that this complex is exact at $\Gamma(X,\mathcal{F})$ and $C^0(\mathfrak{U},\mathcal{F})$.

The higher Čech cohomology groups generally depend on the cover $\mathfrak{U}$, but are not completely unrelated for different covers. What follows is borrowed from chapter 2 section 10 of [BT82].

Definition. *Given covers $\mathfrak{U} = \{U_i\}_{i\in I}$ and $\mathfrak{V} = \{V_j\}_{j\in J}$, we say $\mathfrak{V}$ is a* ***refinement*** *of $\mathfrak{U}$ (we write $\mathfrak{U} < \mathfrak{V}$) if there exists a map $\phi : J \to I$ such that for all $j \in J$, $V_j \subset U_{\phi(j)}$.*

Given such a map ϕ, we can construct maps,

$$
\begin{gathered}
\phi^\# : C^p(\mathfrak{U},\mathcal{F}) \to C^p(\mathfrak{V},\mathcal{F}) \\
\alpha \mapsto \phi^\#\alpha \\
\text{where } (\phi^\#\alpha)_{i_0\dots i_p} = \alpha_{\phi(i_0)\dots\phi(i_p)}\big|_{V_{i_0\dots i_p}}
\end{gathered}
$$

It follows directly from definitions that $\phi \circ d = d \circ \phi$, and hence $\phi^\# : C^\bullet(\mathfrak{U},\mathcal{F}) \to C^\bullet(\mathfrak{V},\mathcal{F})$ is a morphism of cochain complexes. In particular, this induces a map from Čech cohomology w.r.t. $\mathfrak{U}$ to the Čech cohomology w.r.t. $\mathfrak{F}$.

Lemma 2.2. *The map induced on the cohomologies does not depend on the choice of the map ϕ. In particular, given covers $\mathfrak{U}$ and $\mathfrak{V}$ with $\mathfrak{U} < \mathfrak{V}$ there are a well-defined maps $f_{\mathfrak{U}\mathfrak{V}} : \check{H}^\star(\mathfrak{U}, \mathcal{F}) \to \check{H}^\star(\mathfrak{V}, \mathcal{F})$, making the cohomologies $\{\check{H}^\star(\mathfrak{U}, \mathcal{F})\}_{\mathfrak{U}}$ into a directed system of groups indexed by covers $\mathfrak{U}$ ordered by refinement.*

Proof. We show this by constructing a chain homotopy between maps $\phi^\#$ and $\psi^\#$ where $\phi : J \to I$ and $\psi : J \to I$ are such that $V_j \subset U_{\phi(j)} \cap U_{\psi(j)}$ for all $j \in J$. Consider the map:

$$K : C^p(\mathfrak{U}, \mathcal{F}) \to C^{p-1}(\mathfrak{V}, \mathcal{F})$$
$$\alpha \mapsto K\alpha$$
$$\text{where } (K\alpha)_{j_0 \dots j_{p-1}} = \sum_{k=0}^{p-1} (-1)^k \alpha_{\phi(j_0)\dots\phi(j_k)\psi(j_k)\dots\phi(j_{p-1})}$$

Then we have,

$$\begin{aligned}
((Kd)\alpha)_{j_0\dots j_p} &= \sum_{k=0}^{p} (-1)^k (d\alpha)_{\phi(j_0)\dots\phi(j_k)\psi(j_k)\dots\phi(j_p)} \\
&= \sum_{k=0}^{p} (-1)^k \left(\sum_{l=0}^{k} (-1)^l \alpha_{\phi(j_0)\dots\widehat{\phi(j_l)}\dots\phi(j_k)\psi(j_k)\dots\psi(j_p)} + \sum_{l=k}^{p} (-1)^{l+1} \alpha_{\phi(j_0)\dots\phi(j_k)\psi(j_k)\dots\widehat{\psi(j_l)}\dots\psi(j_p)} \right) \\
&= \sum_{k=0}^{p} (-1)^k \sum_{l=k}^{p} (-1)^{l+1} \alpha_{\phi(j_0)\dots\phi(j_k)\psi(j_k)\dots\widehat{\psi(j_l)}\dots\psi(j_p)} + \sum_{k=0}^{p} (-1)^{k-1} \sum_{l=0}^{k} (-1)^{l+1} \alpha_{\phi(j_0)\dots\widehat{\phi(j_l)}\dots\phi(j_k)\psi(j_k)\dots\psi(j_p)} \\
((dK)\alpha)_{j_0\dots j_p} &= \sum_{k=0}^{p} (-1)^k (K\alpha)_{j_0\dots\widehat{j_k}\dots j_p} \\
&= \sum_{k=0}^{p} (-1)^k \left(\sum_{l=0}^{k-1} (-1)^l \alpha_{\phi(j_0)\dots\phi(j_l)\psi(j_l)\dots\widehat{\psi(j_k)}\dots\psi(j_p)} + \sum_{l=k+1}^{p} (-1)^{l-1} \alpha_{\phi(j_0)\dots\widehat{\phi(j_k)}\dots\phi(j_l)\psi(j_l)\dots\psi(j_p)} \right) \\
&= \sum_{l=0}^{p-1} (-1)^l \sum_{k=l+1}^{p} (-1)^k \alpha_{\phi(j_0)\dots\phi(j_l)\psi(j_l)\dots\widehat{\psi(j_k)}\dots\psi(j_p)} + \sum_{l=1}^{p} (-1)^{l-1} \sum_{k=0}^{l-1} (-1)^k \alpha_{\phi(j_0)\dots\widehat{\phi(j_k)}\dots\phi(j_l)\psi(j_l)\dots\psi(j_p)} \\
&= \sum_{k=0}^{p-1} (-1)^k \sum_{l=k+1}^{p} (-1)^l \alpha_{\phi(j_0)\dots\phi(j_k)\psi(j_k)\dots\widehat{\psi(j_l)}\dots\psi(j_p)} + \sum_{k=1}^{p} (-1)^{k-1} \sum_{l=0}^{k-1} (-1)^l \alpha_{\phi(j_0)\dots\widehat{\phi(j_l)}\dots\phi(j_k)\psi(j_k)\dots\psi(j_p)} \\
((Kd + dK)\alpha)_{j_0\dots j_p} &= (-1)^p(-1)^{p+1} \alpha_{\phi(j_0)\dots\phi(j_p)} + \sum_{k=0}^{p-1} (-1)^k (-1)^{k+1} \alpha_{\phi(j_0)\dots\phi(j_k)\psi(j_{k+1})\dots\psi(j_p))} \\
&\quad + \alpha_{\psi(j_0)\dots\psi(j_p)} + \sum_{k=1}^{p} (-1)^{k-1}(-1)^{k+1} \alpha_{\phi(j_0)\dots\phi(j_{k-1})\psi(j_k)\dots\psi(j_p)} \\
&= \alpha_{\psi(j_0)\dots\psi(j_p)} - \alpha_{\phi(j_0)\dots\phi(j_p)} \\
&= (\psi^\#\alpha)_{j_0\dots j_p} - (\phi^\#\alpha)_{j_0\dots j_p}
\end{aligned}$$

Therefore $\psi^\# - \phi^\# = Kd + dK$, and hence K is a chain homotopy between $\phi^\#$ and $\psi^\#$. This immediately gives us that the induced maps on the cohomologies are the same. Hence we denote the map induced on cohomologies by any such $\phi^\#$ to be $f_{\mathfrak{U}\mathfrak{V}}$.
The claim about directed systems is obvious from the following facts:

- $\mathfrak{U} < \mathfrak{U}$, where ϕ can be taken to be identity map id_I on the indexing set of $\mathfrak{U}$. It is clear that $\mathrm{id}_I^\#$ induces the identity map on the Čech cohomology groups, and hence $f_{\mathfrak{U}\mathfrak{U}}$ is the identity map.
- If there are covers $\mathfrak{U}$, $\mathfrak{V}$ and $\mathfrak{W}$ with indexing sets I, J and K, such that $\mathfrak{U} < \mathfrak{V}$ and $\mathfrak{V} < \mathfrak{W}$, and maps $\phi : J \to I$ and $\psi : K \to J$ are such that $V_j \subset U_{\phi(j)}$ for all $j \in J$ and $W_k \subset V_{\psi(k)}$ for all $k \in K$. Then trivially, we have $W_k \subset U_{(\phi\circ\psi)(k)}$ for all $k \in K$ which gives $\mathfrak{U} < \mathfrak{W}$. It is easy to see that $(\phi \circ \psi)^\# = \psi^\# \circ \phi^\#$. This in turn shows that the $f_{\mathfrak{U}\mathfrak{W}} = f_{\mathfrak{V}\mathfrak{W}} \circ f_{\mathfrak{U}\mathfrak{V}}$.

$\square$

Definition. *The* ***Čech cohomology*** $\check{H}^\star(X,\mathcal{F})$ ***of X with values in*** $\mathcal{F}$ *is defined to be the direct limit of the directed system mentioned in the previous lemma, i.e.,*

$$\check{H}^p(X,\mathcal{F}) = \varinjlim_{\mathfrak{U}} \check{H}^p(\mathfrak{U},\mathcal{F})$$

It follows almost immediately from lemma 2.1 and looking at the maps in the directed system of the zeroth Čech cohomologies w.r.t. covers that $\check{H}(X,\mathcal{F}) \cong \Gamma(X,\mathcal{F})$. In general it can be difficult to compute the Čech cohomology groups, but in many cases, it suffices to compute the cohomology w.r.t. a nice enough cover.
Now we'll set up some tools to compute the Čech cohomology using resolutions. The following slightly generalises ideas from chapter 2 section 8 of [BT82].

Definition. *We call a sheaf* $\mathcal{A}$ ***acyclic*** *w.r.t. the cover* $\mathfrak{U}$ *if* $\check{H}^i(\mathfrak{U},\mathcal{A}) = 0$ *for* $i > 0$.

Definition. *Given a sheaf* $\mathcal{F}$*, a complex of sheaves of the form*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{A}^1 \longrightarrow \mathcal{A}^2 \longrightarrow \mathcal{A}^3 \longrightarrow \dots$$

is called a ***resolution w.r.t.*** $\mathfrak{U}$ *if for all* $p \geq 0$ *and all sequences of indices* $i_0 < \cdots < i_p$*, the sequence*

$$0 \longrightarrow \mathcal{F}(U_{i_0\dots i_p}) \longrightarrow \mathcal{A}^1(U_{i_0\dots i_p}) \longrightarrow \mathcal{A}^2(U_{i_0\dots i_p}) \longrightarrow \mathcal{A}^3(U_{i_0\dots i_p}) \longrightarrow \dots$$

Moreover, we call this an ***acyclic resolution w.r.t.*** $\mathfrak{U}$ *if* $\mathcal{A}^i$ *are acyclic w.r.t.* $\mathfrak{U}$ *for all* $i \geq 0$.

Before we actually use these definitions, first a small discussion on double complexes. By a double (cochain) complex of abelian groups (A-modules), we mean a collection $C^{\bullet,\bullet}$ of abelian groups (A-modules) indexed by pairs of integers, and maps $d_v^{p,q} : C^{p,q} \to C^{p,q+1}$ and $d_h^{p,q} : C^{p,q} \to C^{p+1,q}$ (v and h stand for vertical and horizontal respectively), such that $(C^{\bullet,q}, d_h^{\bullet,q})$ and $(C^{p,\bullet}, d_v^{p,\bullet})$ are (cochain) complexes and $d_h^{p,q+1} \circ d_v^{p,q} = d_v^{p+1,q} \circ d_h^{p,q}$ for all $p, q \in \mathbb{Z}$. To make it easier to see what this means, consider the diagram:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\dots \arrow[r, "d_h"] & C^{p-1,q+1} \arrow[r, "d_h"] \arrow[u, "d_v"] & C^{p,q+1} \arrow[r, "d_h"] \arrow[u, "d_v"] & C^{p+1,q+1} \arrow[r, "d_h"] \arrow[u, "d_v"] & \dots \\
\dots \arrow[r, "d_h"] & C^{p-1,q} \arrow[r, "d_h"] \arrow[u, "d_v"] & C^{p,q} \arrow[r, "d_h"] \arrow[u, "d_v"] & C^{p+1,q} \arrow[r, "d_h"] \arrow[u, "d_v"] & \dots \\
\dots \arrow[r, "d_h"] & C^{p-1,q-1} \arrow[r, "d_h"] \arrow[u, "d_v"] & C^{p,q-1} \arrow[r, "d_h"] \arrow[u, "d_v"] & C^{p+1,q-1} \arrow[r, "d_h"] \arrow[u, "d_v"] & \dots \\
 & \vdots \arrow[u, "d_v"] & \vdots \arrow[u, "d_v"] & \vdots \arrow[u, "d_v"] &
\end{tikzcd}$$

where we have omitted the superscripts on the maps $d_v^{\bullet,\bullet}$ and $d_h^{\bullet,\bullet}$ for clarity. Then the above conditions simply say that the rows and columns of this diagram form complexes, and every square in the diagram commutes. We associate a complex to every double complex called its **total complex** $\mathrm{Tot}(C)^\bullet$, defined by:

$$\mathrm{Tot}(C)^n = \bigoplus_{p+q=n} C^{p,q}$$

with the coboundary maps given by:

$$(d^n\alpha)_{p,q} = d_h^{p-1,q}\alpha_{p-1,q} + (-1)^p d_v^{p,q-1}\alpha_{p,q-1}; \quad p+q = n+1$$

where for a cochain $\alpha \in \mathrm{Tot}(C)^n$, we write $\alpha_{p,q}$ for its image under the projection to $C^{p,q}$. To see this actually forms a complex, simply note that,

$$\begin{aligned}((d^{n+1} \circ d^n)\alpha)_{p,q} &= d_h^{p-1,q}(d^n\alpha)_{p-1,q} + (-1)^p d_v^{p,q-1}(d^n\alpha)_{p,q-1} \\ &= d_h^{p-1,q}(d_h^{p-2,q}\alpha_{p-2,q} + (-1)^{p-1}d_v^{p-1,q-1}\alpha_{p-1,q-1}) + (-1)^p d_v^{p,q-1}(d_h^{p-1,q-1}\alpha_{p-1,q-1} + (-1)^p d_v^{p,q-2}\alpha_{p,q-2})) \\ &= (d_h^{p-1,q} \circ d_h^{p-2,q})\alpha_{p-2,q} + (-1)^{p-1}(d_h^{p-1,q} \circ d_v^{p-1,q-1} - d_v^{p,q-1} \circ d_h^{p-1,q-1})\alpha_{p-1,q-1} + (d_h^{p,q-2} \circ d_h^{p,q-2})\alpha_{p,q-2} \\ &= 0\end{aligned}$$

where the first, second and third term in the last expression vanish due to the rows forming a complex, commutativity of the squares and the columns forming a complex respectively.

Remark: What we have defined here is generally denoted as $\mathrm{Tot}^{\oplus}(C)^\bullet$. One can also define a complex $\mathrm{Tot}^{\prod}(C)^\bullet$ with $\mathrm{Tot}^{\prod}(C)^n = \prod_{p+q=n} C^{p,q}$ and the same coboundary maps, with the exact same proof showing that this forms a complex. But we are only going to be working with in double complexes which are **bounded in the first quadrant**, i.e. $C^{p,q} = 0$ if $p < 0$ or $q < 0$. In this case, the diagonals $p + q = n$ contain only finitely many non-trivial groups (A-modules) so both these definitions coincide.

For a double complex $C^{\bullet,\bullet}$ bounded in the first quadrant, consider the augmented complex formed by adding a column of degree -1 given by $\ker d_h^{0,\bullet} : C^{0,\bullet} \to C^{1,\bullet}$. This looks like:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & K^2 \arrow[u, "e^2"] \arrow[r, "\epsilon^2"] & C^{0,2} \arrow[u, "d_v"] \arrow[r, "d_h"] & C^{1,2} \arrow[u, "d_v"] \arrow[r, "d_h"] & \dots \\
0 \arrow[r] & K^1 \arrow[u, "e^1"] \arrow[r, "\epsilon^1"] & C^{0,1} \arrow[u, "d_v"] \arrow[r, "d_h"] & C^{1,1} \arrow[u, "d_v"] \arrow[r, "d_h"] & \dots \\
0 \arrow[r] & K^0 \arrow[u, "e^0"] \arrow[r, "\epsilon^0"] & C^{0,0} \arrow[u, "d_v"] \arrow[r, "d_h"] & C^{1,0} \arrow[u, "d_v"] \arrow[r, "d_h"] & \dots
\end{tikzcd}$$

where $(K^\bullet, e^\bullet)$ is the kernel of the morphism of complexes $d_h^{0,\bullet}$ and $\epsilon^\bullet$ is the inclusion of the complex $K^\bullet$ into $C^{0,\bullet}$. Then we have:

Lemma 2.3. *If the augmented complex above has exact rows, then the cohomology of the total complex* $\mathrm{Tot}(C)^\bullet$ *is isomorphic to the cohomology of the the complex* $K^\bullet$.

Proof. First observe that there is an obvious map of complexes from $i^\bullet : C^{0,\bullet} \to \mathrm{Tot}(C)^\bullet$, where $i^\bullet$ is the canonical map from $C^{0,\bullet} \to \oplus_{p+q=\bullet} C^{p,q}$. We define the map $r^\bullet = i^\bullet \circ \epsilon^\bullet : K^\bullet \to \mathrm{Tot}(C)^\bullet$. We will show that the induced map on the cohomologies $r^\star : h^\star(K^\bullet) \to h^\star(\mathrm{Tot}(C)^\bullet)$ are isomorphisms.

To show this first we will prove the following lemma: if $\alpha = (\alpha_{p,q})_{p+q=n}$ is a cochain in the total complex such that $(d^n\alpha)_{p,q} = 0$ for all $p > 1$, then there exists a $\beta_{0,n} \in C^{0,n}$ such that the cochain $\beta = (\beta_{p,q})_{p+q=n}$ defined as $\beta_{p,q} = 0$ for $p \neq 0$ differs from α by a coboundary.
We prove this by inducting on the the largest p_m such that $\alpha_{p_m,q} \neq 0$. Firstly, we know such a p_m exists since our double complex is bounded. If $p_m = 0$, we are trivially done with $\beta_{0,n} = \alpha_{0,n}$. Set $q_m = n - p_m$. If $p_m > 0$, note that,

$$\begin{aligned}0 = (d\alpha)_{p_m+1,q_m} &= d_h^{p_m,q_m}\alpha_{p_m,q_m} + (-1)^{p_m} d_v^{p_m+1,q_m-1}\alpha_{p_m+1,q_m-1} \\ &= d_h^{p_m,q_m}\alpha_{p_m,q_m}\end{aligned}$$

since $\alpha_{p_m+1,q_m-1} = 0$. Then by exactness of the rows at C^{p_m,q_m}, $\alpha_{p_m,q_m} = d_h^{p_m-1,q_m}\gamma_{p_m-1,q_m}$ for some $\gamma_{p_m-1,q_m} \in C^{p_m-1,q_m}$. Consider the cochain $\gamma = (\gamma_{p,q})_{p+q=n-1}$, given by $\gamma_{p,q} = 0$ for $p \neq p_m - 1$. Then $(\alpha - d^{n-1}\gamma)_{p,q} = 0$ for $p > p_m$ since $\alpha_{p,q} = 0$ for $p > p_m$ and $\gamma_{p,q} = 0$ for $p \geq p_m$. But we also have,

$$(\alpha - d^{n-1}\gamma)_{p_m,q_m} = \alpha_{p_m,q_m} - (d_h^{p_m-1,q_m}\gamma_{p_m-1,q_m} + (-1)^{p_m} d_v^{p_m,q_m-1}\gamma_{p_m,q_m-1}) = 0$$

We have found a cochain $\alpha' = \alpha - d^{n-1}\gamma$ which differs from α by the coboundary $d^{n-1}\gamma$, such that the largest p for which $\alpha'_{p,q} \neq 0$ is less than p_m. Therefore, we are done by induction.

Now to show that r^* is an isomorphism.

- r^* is surjective. By the above lemma, for every cohomology class in the total complex contains a representative of the form $\beta = (\beta_{p,q})_{p+q=n}$ such that $\beta_{p,q} = 0$ for $p \neq 0$. We have $0 = (d^n\beta)_{1,n} = d_h^{0,n}\beta_{0,n} + (-1)^1 d_v^{1,n-1}\beta_{1,n-1} = d_h^{0,n}\beta_{0,n}$. Then by exactness of the rows at $C^{0,n}$, $\beta_{0,n} = \epsilon_n c$ for some $c \in K^n$. Moreover $(\epsilon^{n+1} \circ e_n)c = (d_v^{0,n+1} \circ \epsilon_n)c = d_n^{0,n+1}\beta_{0,n} = 0$, which implies $e_n c = 0$ due to the injectivity of ϵ^{n+1}. Therefore c represents a cohomology class of $K^\bullet$ and $r^n c = \beta$, which gives us that r^* is surjective.

- r^* is injective. If c is a $K^\bullet$ n-cocycle which maps to a coboundary under $r^\bullet$, i.e. $r^n c = d^{n-1}\alpha$ for an $(n-1)$-cochain α of the total complex. It is easy to see that $(d^{n-1}\alpha)_{p,q} = (r^n c)_{p,q} = 0$ for $p \geq 1$, so by the above lemma, there exists a $\beta = (\beta_{p,q})_{p+q=n-1}$ such that $\beta_{p,q} = 0$ for $p \neq 0$ which differs from α by a coboundary. Moreover $0 = (d^{n-1}\alpha)_{1,n-1} = (d^{n-1}\beta)_{1,n-1} = d_h^{0,n-1}\beta_{0,n-1} + (-1)^1 d_v^{1,n-2}\beta_{1,n-2} = d_h^{0,n-1}\beta_{0,n-1}$. Therefore by exactness of rows at $C^{0,n-1}$, there exists a $c' \in K^{n-1}$ such that $\epsilon^{n-1}c' = \beta_{0,n-1}$. Finally since $(\epsilon^n \circ e_{n-1})c' = (d_v^{0,n} \circ \epsilon_{n-1})c' = d_v^{0,n-1}\beta_{0,n-1} = (d^{n-1}\beta)_{0,n} = (d^{n-1}\alpha)_{0,n} = (r^n c)_{0,n} = \epsilon_n c$, and ϵ^n is injective, $c = e_{n-1}c'$ and hence is a coboundary. This shows that r^* is injective.

$\square$

We can also augment our double complex by adding a row of degree -1 given by $\ker d_v^{\bullet,0} : C^{\bullet,0} \to C^{\bullet,1}$. This looks like:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
C^{0,1} \arrow[r, "d_h"] \arrow[u, "d_v"] & C^{1,1} \arrow[r, "d_h"] \arrow[u, "d_v"] & C^{2,1} \arrow[r, "d_h"] \arrow[u, "d_v"] & \dots \\
C^{0,0} \arrow[r, "d_h"] \arrow[u, "d_v"] & C^{1,0} \arrow[r, "d_h"] \arrow[u, "d_v"] & C^{2,0} \arrow[r, "d_h"] \arrow[u, "d_v"] & \dots \\
L^0 \arrow[r, "f^0"] \arrow[u, "\delta^0"] & L^1 \arrow[r, "f^1"] \arrow[u, "\delta^1"] & L^2 \arrow[r, "f^2"] \arrow[u, "\delta^2"] & \dots \\
0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \dots
\end{tikzcd}$$

where $(L^\bullet, f^\bullet)$ is the kernel of the morphism of complexes $d_v^{\bullet,0}$ and $\delta^\bullet$ is the inclusion of the complex $L^\bullet$ into $C^{\bullet,0}$. By essentially the same argument as for the proof of lemma 2.3, we can prove:

Lemma 2.4. *If the augmented complex above has exact columns, then the cohomology of the total complex* $\mathrm{Tot}(C)^\bullet$ *is isomorphic to the cohomology of the the complex* $L^\bullet$*.*

Proof. Omitted $\square$

Coming back to Čech cohomology, for a cover $\mathfrak{U}$ and a complex of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{A}^1 \longrightarrow \mathcal{A}^2 \longrightarrow \mathcal{A}^3 \longrightarrow \dots$$

Consider the double complex $C^{p,q} = C^p(\mathfrak{U}, \mathcal{A}^q)$, where the row maps are the Čech coboundary maps and the column maps are the ones induced on the sections over open sets by the morphism of sheaves in the given complex. Then we have the following theorem,

Theorem 2.5. *1. If* $\mathcal{A}^q$ *are acyclic w.r.t the cover* $\mathfrak{U}$*, then* $h^\star(\Gamma(X, \mathcal{A}^\bullet)) \cong h^\star(C^\bullet(\mathfrak{U}, \mathcal{A}^\bullet))$*.*

2. If the complex,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{A}^1 \longrightarrow \mathcal{A}^2 \longrightarrow \mathcal{A}^3 \longrightarrow \dots$$

is a resolution w.r.t. the cover $\mathfrak{U}$*, then* $\check{H}^\star(\mathfrak{U}, \mathcal{F}) \cong h^\star(C^\bullet(\mathfrak{U}, \mathcal{A}^\bullet))$*.*

Proof.

1. Consider the augmented complex as in lemma 2.3 for the double complex $C^p(\mathfrak{U}, \mathcal{A}^q)$. It follows easily from 2.1 that the complex $K^\bullet = \Gamma(X, \mathcal{A}^\bullet)$. Moreover, the condition that A^q are acyclic w.r.t $\mathfrak{U}$ is exactly the condition that the rows of this augmented complex are exact. Then the result immediately follows from lemma 2.3.

2. Consider the augmented complex as in lemma 2.4 for the double complex $C^p(\mathfrak{U}, \mathcal{A}^q)$. Since the given complex is a resolution w.r.t. $\mathfrak{U}$, we have,

$$0 \longrightarrow \mathcal{F}(U_{i_0 \dots i_p}) \longrightarrow \mathcal{A}^1(U_{i_0 \dots i_p}) \longrightarrow \mathcal{A}^2(U_{i_0 \dots i_p}) \longrightarrow \mathcal{A}^3(U_{i_0 \dots i_p}) \longrightarrow \dots$$

for all increasing sets of indices $i_0 < \dots < i_p$. Taking products this shows us both that the complex $L^\bullet = C^\bullet(\mathfrak{U}, \mathcal{F})$ and that the columns of the augmented complex are exact. Then the result immediately follows from lemma 2.4.

□

Corollary 2.6. *If the given complex is an acyclic resolution w.r.t* $\mathfrak{U}$*, then* $\check{H}^\star(\mathfrak{U}, \mathcal{F}) \cong h^\star(\Gamma(X, \mathcal{A}^\bullet))$.

Finally, we define an important class of sheaves and quote and important theorem about them, which we will use later on.

Definition. *A sheaf* $\mathcal{F}$ *on* X *is said to be* ***flasque****, if all its restriction maps are surjective.*

Theorem 2.7. *Given a flasque sheaf* $\mathcal{F}$ *and any cover* $\mathfrak{U}$ *on* X, $\check{H}^p(\mathfrak{U}, \mathcal{F}) = 0$ *for all* $p > 0$.

Proof. See Chapter 3, Proposition 4.3 of [Har77]. □

3 de Rham cohomology and the Generalized Mayer-Vietoris Sequence

Throughout this section, X will be a smooth manifold unless stated otherwise.

Definition. *The* ***sheaf of differential*** q***-forms*** Ω^q_X *is defined to be the sheaf of smooth sections of the qth exterior power of the cotangent bundle,* $\bigwedge^r(T^*X) \to X$*. Note that since this is a real vector bundle, this is a sheaf of with values in* $\mathfrak{Mod}(\mathbb{R})$*, i.e. the category of real vector spaces.*
The (sheafified) ***de Rham complex*** *is the complex of sheaves given by*

$$0 \longrightarrow \Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \dots$$

where the coboundary maps are given by taking exterior derivatives of differential forms over each open set.

To differentiate the Čech coboundary from the exterior derivative map, we will use δ for the Čech coboundary for the rest of this document.
The usual de Rham complex can be obtained by applying the global sections functor to the sheafified de Rham complex.

Definition. *The* ***de Rham cohomology of*** X*,* $H^\star_{dR}(X)$ *is given by the cohomology of the complex of global sections of the de Rham complex, i.e.* $H^\star_{dR}(X) = h^\star(\Gamma(X, \Omega^\bullet_X))$.

The de-Rham cohomology of Euclidean space is given by:

Theorem 3.1. *(Poincaré Lemma of de Rham cohomology)*

$$H^\star_{dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & ; \star = 0 \\ 0 & ; \textit{otherwise} \end{cases}$$

A proof can be found in any standard text on manifolds.

Now we state and prove the Generalized Mayer-Vietoris Sequence:

Theorem 3.2. *(The Generalized Mayer-Vietoris Sequence)*
For any cover $\mathfrak{U}$ of X, the augmented Čech complex for $\Omega^\star_X$, i.e.,

$$0 \longrightarrow \Gamma(X, \Omega^\star_X) \longrightarrow C^0(\mathfrak{U}, \Omega^\star_X) \xrightarrow{\delta} C^1(\mathfrak{U}, \Omega^\star_X) \xrightarrow{\delta} \dots$$

is exact. Equivalently, $\Omega^\star_X$ are acyclic w.r.t any cover $\mathfrak{U}$.

Proof. We show this by constructing a chain homotopy between the identity and the zero maps from this complex to itself. Since chain homotopic maps induce the same maps on cohomologies, the identity map on the cohomologies must be the same as the zero map, i.e. the cohomologies vanish and the complex is exact. Consider a partition of unity $\{\rho_i\}_{i\in I}$ subordinate to the cover $\mathfrak{U} = \{U_i\}_{i\in I}$. Consider the map:

$$K : C^p(\mathfrak{U}, \Omega^\star_X) \to C^{p-1}(\mathfrak{U}, \Omega^\star_X)$$
$$\omega \mapsto K\omega$$
$$\text{where } (K\omega)_{i_0 \dots i_{p-1}} = \sum_{i\in I} \rho_i \omega_{i i_0 \dots i_{p-1}}$$

Then we have,

$$\begin{aligned}
((\delta K + K\delta)\omega)_{i_0\dots i_p} &= \sum_{k=0}^{p} (-1)^k (K\omega)_{i_0\dots \widehat{i_k} \dots i_p} + \sum_{i\in I} \rho_i (\delta\omega)_{i i_0 \dots i_p} \\
&= \sum_{k=0}^{p} (-1)^k \sum_{i\in I} \rho_i \omega_{i i_0 \dots \widehat{i_k} \dots i_p} + \sum_{i\in I} \rho_i \omega_{i_0\dots i_p} + \sum_{i\in I} \sum_{k=0}^{p} (-1)^{k+1} \rho_i \omega_{i i_0 \dots \widehat{i_k} \dots i_{p-1}} \\
&= \left(\sum_{i\in I} \rho_i \right) \omega_{i_0\dots i_p} = \omega_{i_0\dots i_p}
\end{aligned}$$

Hence $\delta K + K\delta = \text{id}$, and we are done. □

Remark: If our cover consists of just two open sets, this reduces to the short exact sequence of complexes,

$$0 \longrightarrow \Omega^\star_X(X) \longrightarrow \Omega^\star_X(U) \oplus \Omega^\star_X(V) \longrightarrow \Omega^\star_X(U \cap V) \longrightarrow 0$$

whose long exact sequence gives us the usual Mayer-Vietoris sequence.

Definition. *Given a topological space X, and an abelian group (A module) M, the* ***locally constant sheaf*** *M_X* ***associated to*** *M is the sheaf of locally constant functions to M, i.e.*

$$M_X(U) = \{f : U \to M \mid f \text{ is locally constant}\}$$

where the restriction map r_{UV}, restricts the domain of the a function f from U to V.
When X is manifold and $M = \mathbb{R}$, the sheaf $\mathbb{R}_X$ is simply the ***sheaf of locally constant (real-valued) functions on X****.*

Note that Ω^0_X is simply the sheaf of real-valued smooth functions on X. If a smooth function f is such that $df = 0$, then f is locally constant. Therefore, the kernel of $d : \Omega^0_X \to \Omega^1_X$ is exactly $\mathbb{R}_X$. Therefore we have a complex of sheaves,

$$0 \longrightarrow \mathbb{R}_X \longrightarrow \Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \dots$$

where Ω^q_X are acyclic w.r.t any cover $\mathfrak{U}$ of X. Poincaré Lemma says that if $X = \mathbb{R}^n$ and the cover $\mathfrak{U}$ consists of just one element, i.e. the entire space, then the above complex is a resolution w.r.t. this cover. More generally, for any manifold X, if we have a cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of X such that for any finite sequence of indices $i_0 < \dots < i_p$, $U_{i_0\dots i_p} \cong \mathbb{R}^n$, then the above complex is an acyclic resolution w.r.t. this cover. We call such covers good covers. Turns out:

Theorem 3.3. *Every manifold admits a good cover. Moreover, such covers are cofinal in the directed system of covers ordered by refinement.*

Proof. See Theorem 5.1 and the discussion afterwards in [BT82] $\square$

We can immediately conclude:

Theorem 3.4. *For any manifold X, and a good cover $\mathfrak{U}$ of X,*

$$H^\star_{dR}(X) \cong \check{H}^\star(\mathfrak{U}, \mathbb{R}_X) \cong \check{H}^\star(X, \mathbb{R}_X)$$

Proof. The second isomorphism follows from the cofinality claim of theorem 3.3. The first isomorphism follows from the discussion preceding theorem 3.3 and corollary 2.6. $\square$

4 Singular Cohomology

Let X now be a locally contracible topological space, and M be an abelian group (A-module).

Definition. *The **sheaf of singular q-chains with values in** M, $\mathcal{C}^p(X, M)$ is given by, $\mathcal{C}^p(X, M)(U) = C^p(U, M)$, the group (module) of singular p-cochains on the subspace U, with the restriction maps being composition of cochains on U with the pushforward of singular p-chains from a smaller open set V to U.*
*The (sheafified) **singular cochain complex of** X **with values in** M is the complex:*

$$0 \longrightarrow \mathcal{C}^0(X, M) \xrightarrow{d} \mathcal{C}^1(X, M) \xrightarrow{d} \mathcal{C}^2(X, M) \xrightarrow{d} \dots$$

where d is defined as the usual singular coboundary map over each open set.

The usual singular cochain complex can be obtained by taking global sections of the sheafified complex.

Definition. *The **singular cohomology of** X **with values in** M, $H^\star(X, M)$ is given by the cohomology of the complex of global sections of the singular cochain complex, i.e. $H^\star(X, M) = h^\star(\Gamma(X, \mathcal{C}^\bullet(X, M)))$.*

For contractible spaces:

Theorem 4.1. *(Poincaré Lemma for singular cohomology)*
If X is contractible,

$$H^\star(X, M) = \begin{cases} M & ; \star = 0 \\ 0 & ; \textit{otherwise} \end{cases}$$

Proof. A proof can be found in any standard text on algebraic topology. $\square$

Note that $\mathcal{C}^0(X, M)$ is just the sheaf of set-theoretic maps from X to M, with $\ker d : \mathcal{C}^0(X, M) \to \mathcal{C}^0(X, M)$ being the sheaf of those maps from $X \to M$ which take the same value on the boundary of singular 1-chains, i.e. constant on path-components. Since X is locally contractible, path-components are the same as connected components, therefore the kernel is exactly the locally constant sheaf M_X. Therefore we have a complex of sheaves,

$$0 \longrightarrow M_X \longrightarrow \mathcal{C}^0(X, M) \longrightarrow \mathcal{C}^1(X, M) \longrightarrow \mathcal{C}^2(X, M) \longrightarrow \dots$$

The sheaves $\mathcal{C}^\star(X, M)$ are trivially flasque, since any cochain on an open subset can be extended by zero to a cochain on the entire space which restricts to the cochain we started with. Hence by theorem 2.7, $C^\star(X, M)$ are acyclic. Consider the case X is a manifold. Since $\mathbb{R}^n$ is contractible, by Poincaré lemma for singular cohomology, the above complex is acyclic w.r.t. a good cover of X. By exactly the same argument as in theorem 3.4, we get,

Theorem 4.2. *For any manifold X, and a good cover $\mathfrak{U}$ of X,*

$$H^\star(X, M) \cong \check{H}^\star(\mathfrak{U}, M_X) \cong \check{H}^\star(X, M_X)$$

Corollary 4.3. *(The de Rham isomorphism)*

$$H^\star(X, \mathbb{R}) \cong H^\star_{dR}(X)$$

Proof. Follows immediately from theorems 3.4 and 4.2 for $M = \mathbb{R}$. $\square$

References

[BT82] Raoul Bott and Loring W. Tu. *Differential Forms in Algebraic Topology*. Springer New York, 1982.

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