Number Fields Races

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$\pi_K(x)$ vs. $\mathrm{li}(x)$

In 1903, Landau proved the Prime Ideal Theorem,

$$\psi_K(x) \sim x \qquad\qquad \pi_K(x) \sim \mathrm{li}(x)$$

where,

$$\psi_K(x) = \sum_{N(\mathfrak{p}^k) \leq x} \log N(\mathfrak{p}) \qquad\qquad \pi_K(x) = \sum_{N(\mathfrak{p}) \leq x} 1$$

Moreover, in 1918, assuming a slightly weakened form of the GRH (allowing for real zeroes not on the critical line) for $\zeta_K(s)$, he proved,

$$\psi_K(x) - x + \sum_{\gamma=0} \frac{x^\rho}{\rho} = \Omega_\pm\left(\sqrt{x} \log\log\log x\right)$$

$$\pi_K(x) - \mathrm{li}(x) + \sum_{\gamma=0} \mathrm{li}(x^\rho) = \Omega_\pm\left(\frac{\sqrt{x}\log\log\log x}{\log x}\right)$$

$\pi_K(x)$ vs. $\mathrm{li}(x)$

In 1982, Staś, taking only number fields K such that $\zeta_K(s)$ has no real zeroes in $(0, 1)$, showed unconditionally that,

$$\max_{1 \leq x \leq T} (\psi_K(s) - x) > T^\beta \exp\left(-15 \frac{\log T}{\sqrt{\log \log T}}\right) \quad T >> 0$$

$$\min_{1 \leq x \leq T} (\psi_K(s) - x) < -T^\beta \exp\left(-15 \frac{\log T}{\sqrt{\log \log T}}\right) \quad T >> 0$$

where β is the real part of an arbitrary zero of $\zeta_K(s)$. Furthermore, in 1988, Kaczorowski and Staś showed that for such K and any $0 < \epsilon < 1$, the number of sign changes $V_K(T)$ of $\psi_K(x) - x$ has the following lower bound,

$$V_K(T) \geq (1 - \epsilon)\frac{\gamma_K}{\pi} \log T \quad T > C(\epsilon)$$

where γ_K is the imaginary part of the lowest zero of $\zeta_K(s)$ in the upper half of the critical strip.

Generalized Ideal Classes

Fix a subset $\mathfrak{f}_\infty$ of real embeddings of K and an ideal $\mathfrak{f}_0$ of the ring of integers $\mathcal{O}_\mathcal{K}$. Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ coprime to $\mathfrak{f}_0$ are called equivalent mod $\mathfrak{f} = (\mathfrak{f}_0, \mathfrak{f}_\infty)$ if there exist $\alpha, \beta \in \mathcal{O}_\mathcal{K}$, such that

$$\alpha \equiv \beta \equiv 1 \pmod{\mathfrak{f}_0}$$

$$\alpha\mathfrak{a} = \beta\mathfrak{b}$$

$$\sigma(\alpha), \sigma(\beta) > 0, \quad \forall \sigma \in \mathfrak{f}_\infty$$

The equivalence classes of ideals coprime to $\mathfrak{f}$ under this relation form a finite group under ideal multiplication called the $\mathfrak{f}$-ideal class group. It is denoted by $\mathcal{H}(\mathfrak{f})$.
If $\mathfrak{f}_\infty = \phi$ and $\mathfrak{f}_0 = \mathcal{O}_K$, then this is just the usual ideal class group.
If $\mathfrak{f}_\infty$ consists of all real embeddings, then this group is called the **ray class group mod** $\mathfrak{f}_0$. We will denote this by $\mathcal{H}(\mathfrak{f}_0)$. If $K = \mathbb{Q}$ then the ray class group mod $a\mathbb{Z}$ is precisely the group of reduced residue classes mod a.

Prime ideals in $\mathfrak{f}$-ideal classes

Let χ be a character on the $\mathfrak{f}$-ideal class group $\mathcal{H}(\mathfrak{f})$. Then one can define the associated Hecke L-function as,

$$\zeta(s, \chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s}$$

where $\chi(\mathfrak{a})$ is just the value of χ at the class of $\mathfrak{a}$ if $\mathfrak{a}$ is coprime to $\mathfrak{f}_0$, otherwise its 0. Again, for ray class groups of $\mathbb{Q}$, these are exactly the Dirichlet L-functions.

For an $\mathfrak{f}$-ideal class $\mathfrak{K}$, consider the functions,

$$\psi(x, \mathfrak{K}) = \sum_{\substack{\mathfrak{p}^k \in \mathfrak{K} \\ N(\mathfrak{p}^k) \leq x}} \log N\mathfrak{p} \qquad \pi(x, \mathfrak{K}) = \sum_{\substack{\mathfrak{p} \in \mathfrak{K} \\ N(\mathfrak{p}) \leq x}} 1$$

It can be shown that,

$$\psi(x, \mathfrak{K}) \sim \frac{1}{|\mathcal{H}(\mathfrak{f})|} x \qquad \pi(x, \mathfrak{K}) \sim \frac{1}{|\mathcal{H}(\mathfrak{f})|} \operatorname{li}(x)$$

Prime ideals in ray classes mod $\mathfrak{f}$

In 1976, Staś and Wiertelak showed the following correspondence between zero-free regions of some Hecke L-functions and order of growth of the difference $\psi(x, \mathfrak{K}) - \psi(x, \mathfrak{K}_0)$: If γ_1 is the supremum of numbers γ for which,

$$\psi(x, \mathfrak{K}) - \psi(x, \mathfrak{K}_0) = O(xe^{-a(\log x)^{\gamma}})$$

for some positive constant a and if γ_2 is the infimum of numbers γ for which

$$\prod_{\chi(\mathfrak{K}) \neq 1} \zeta(s, \chi) \neq 0$$

in the region

$$\sigma > 1 - \frac{b}{(\log |t|)^{\gamma}}, \quad |t| > c$$

for some positive constants b and c. Then

$$\gamma_1 = \frac{1}{1 + \gamma_2}$$

Chebotarev density and the Conjugacy Class Race

We know from the Chebotarev density theorem that for a conjugacy class C of the Galois group G of a Galois extension L/K of number fields, we have,

$$\pi_C(x) \sim \frac{|C|}{|G|} \mathrm{li}(x)$$

where

$$\pi_C(x) = \{\mathfrak{p}, \text{ prime of } K | N(\mathfrak{p}) \leq x, \sigma_{\mathfrak{p}} = C\}$$

where $\sigma_{\mathfrak{p}}$ is the Artin symbol. So one can consider the race between the normalized functions $\frac{|G|}{|C_1|}\pi_{C_1}(x)$ and $\frac{|G|}{|C_2|}\pi_{C_2}(x)$ for distinct conjugacy classes C_1 and C_2.

Biases in the Ideal Class and the Chebotarev races

In his PhD thesis, Nathan Ng applied the ideas from Rubinstein and Sarnak's paper *Chebyshev's Bias* to two prime ideal races,

- Prime ideal counting functions for ideal classes.
- Prime ideal counting functions for conjugacy classes of Galois groups.

Rubinstein and Sarnak assume the Linear Independence hypothesis which in particular implies that Dirichlet L-functions do not vanish at $s = \frac{1}{2}$. But in this setting, the analogous Ideal class and Artin L-functions may have zeroes at $s = \frac{1}{2}$, in fact there are known examples of Artin L-functions with a zero at that point. So, he considers the modified linear independence hypothesis for just positive ordinates of the zeroes.

The Ideal Class Race

Consider the ideal class group $\mathcal{H}_{\mathcal{K}}$. For an ideal class $\mathfrak{a}$, define,

$$sq^{-1}(\mathfrak{a}) = \{\mathfrak{b} \in \mathcal{H}_K | \mathfrak{b}^2 = \mathfrak{a}\}$$

For ideal classes $\mathfrak{a}_1$ and $\mathfrak{a}_2$, N. Ng. derived the following expression,

$$\frac{\log x}{\sqrt{x}}(\pi_{\mathfrak{a}_1}(x) - \pi_{\mathfrak{a}_2}(x))$$
$$= \frac{|sq^{-1}(\mathfrak{a}_2)| - |sq^{-1}(\mathfrak{a}_1)|}{|\mathcal{H}_K|} - \frac{1}{\mathcal{H}_K} \sum_{\chi \in \widehat{\mathcal{H}_K}, \chi \neq 1} (\overline{\chi(\mathfrak{a}_1)} - \overline{\chi(\mathfrak{a}_2)}) \sum_{\gamma_\chi} \frac{x^{i\gamma_\chi}}{\frac{1}{2} + \gamma_\chi} + \text{ small error terms}$$

If for a character χ, $L(\frac{1}{2}, \chi) = 0$, then that causes the sum to have non-oscillating terms which cause a bias. Hence the "bias factor" is given by

$$c(\mathfrak{a}) = c_{sq}(\mathfrak{a}) + c_{\frac{1}{2}}(\mathfrak{a})$$

The Ideal Class Race

where

$$c_{sq}(\mathfrak{a}) = |sq^{-1}(\mathfrak{a})|$$

is the classical Chebyshev bias and

$$c_{\frac{1}{2}}(\mathfrak{a}) = 2 \sum_{\chi \neq 1} \overline{\chi(\mathfrak{a})} n_\chi$$

where n_χ is the order of zero of $L(x, \chi)$ at $s = \frac{1}{2}$, is the bias term caused by the vanishing at $\frac{1}{2}$

Finally, he proved a Central Limit Theorem for multi-way ideal class races in imaginary quadratic fields as discriminant tends to ∞, showing that the biases disappear as the discriminant becomes large.

The Conjugacy Class Race

Similar to the Ideal Class race, for a conjugacy class C of $G = Gal(L/K)$ define,

$$sq^{-1}(C) = \bigcup_{C'^2 \subset C} C'$$

Again, he derived the following expression,

$$\frac{\log x}{\sqrt{x}} \left(\frac{|G|}{|C_1|}\pi_{C_1}(x) - \frac{|G|}{|C_2|}\pi_{C_2}(x) \right)$$
$$= \left(\frac{|sq^{-1}(C_2)|}{|C_2|} - \frac{|sq^{-1}(C_1)|}{|C_1|} \right) - \sum_{\chi \neq 1} (\overline{\chi(C_1)} - \overline{\chi(C_2)}) \sum_{\gamma_\chi} \frac{x^{i\gamma_\chi}}{\frac{1}{2} + \gamma_\chi} + \text{ small error terms}$$

Again, we have the bias factor $c(C)$ given by,

$$c(C) = c_{sq}(C) + c_{\frac{1}{2}}(C), \quad c_{sq}(C) = \frac{|sq^{-1}(C)|}{|C|}, \quad c_{\frac{1}{2}}(C) = 2\sum_{\chi \neq 1} \overline{\chi(C)} n_\chi$$

The Conjugacy Class Race

He further considered an example of Serre of the Galois extension $\mathbb{Q}(\sqrt{\frac{5+\sqrt{5}}{2}\frac{41+\sqrt{5\cdot 41}}{2}})$ of $\mathbb{Q}$ with Galois group isomorphic to the quaternion group H_8 which has an Artin L-function which vanishes at $s = \frac{1}{2}$. Let $C_1 = \{1\}$ and $C_2 = \{-1\}$. Then he computed

$$c_{sq}(C_1) = 2 \qquad c_{sq}(C_2) = 6$$

$$c_{\frac{1}{2}}(C_1) = 4n \qquad c_{\frac{1}{2}}(C_2) = -4n$$

where n is the order of the zero at $s = \frac{1}{2}$ of the Artin L-function mentioned above. Since $n \geq 1$, this shows that the bias due to the zero actually overcomes the classically expected bias. He then verified this by computing,

$$\frac{1}{\log x}\int_2^x 1_{\{x\geq 2 | \pi_{C_2}(x) > \pi_{C_1}(x)\}} \frac{dt}{t} = 0.8454\ldots$$

for $x = 10^8$.

The Conjugacy Class Race

Making a similar computation for the Galois extension $\mathbb{Q}(\sqrt{(2+\sqrt{2})(3+\sqrt{3})})$ of $\mathbb{Q}$, also with Galois group H_8, he found,

$$\frac{1}{\log x}\int_2^x 1_{\{x\geq 2|\pi_{C_1}(x)>\pi_{C_2}(x)\}}\frac{dt}{t} = 0.7391\ldots$$

again for $x = 10^8$, and noted that this suggests non-vanishing for the corresponding Artin L-function at $s = \frac{1}{2}$ for this extension.