The Riemann-Roch Theorem

Devang Agarwal

Abstract: In this article, we present a proof of the Riemann-Roch theorem on smooth irreducible algebraic curves given in the article [Kem77].

1 Introduction

Fix a smooth irreducible algebraic curve C over an algebraically closed field k. For an invertible sheaf $\mathscr{L}$ on C, let $\mathscr{K}(\mathscr{L})$ denote the constant sheaf of rational sections of $\mathscr{L}$, and $\mathscr{P}(\mathscr{L})$ denote the quotient sheaf $\mathscr{K}(\mathscr{L})/\mathscr{L}$. We'd like to show that the sheaf $\mathscr{P}(\mathscr{L})$ is isomorphic to the direct sum of its stalks. Since the question is local, by restricting ourselves to an affine open set $U \subseteq C$ where $\mathscr{L}$ is trivial, we can see this by the lemma:

Lemma 1.1. *Let A be a Dedekind domain and K be its field of fractions. Then the natural map,*

$$K/A \to \bigoplus_m K/A_m$$

is an isomorphism.

Proof. Note that the injectivity of the map is clear from the fact that $\cap_m A_m = A$ for a domain A. To show surjectivity, fix a maximal ideal m_0 of A. Pick $t_1 \in A$ such that $v_{m_0}(t_1) = 1$, where v_m denotes the valuation associated to the maximal ideal m. Consider the finite set $P = \{m \mid v_m(t_1^{-1}) < 0\} - \{m_0\}$. Let $I = \prod_{m \in P} m^{v_m(t_1)}$ and pick $a \in I - m_0 I$. Now note that for $t = a^{-1}t_1$, $v_m(t) \leq 0$ for all $m \neq m_0$, and $v_{m_0} = 1$.
Take a set of representatives S of A/m_0 in A. Then any $c \in K$ has a power series expansion of the form $c = \sum_{n<0} a_n t^{-n} + d$, where the sum is finite, $a_n \in S$ and $d \in A_{m_0}$. Set $c_0 = \sum_{n<0} a_n t^{-n}$. Then c_0 has image c in K/A_{m_0} and 0 in other components, since for $m \neq m_0$, $v_m(c_0) \geq 0$. Hence, the map is surjective. ■

From this we see that $\mathscr{P}(\mathscr{L})$ is a flasque sheaf, and hence has trivial cohomology.
Now pick an effective divisor D on C. We have inclusions $\mathscr{L} \hookrightarrow \mathscr{L}(D) \hookrightarrow \mathscr{K}(\mathscr{L})$. Taking quotients by $\mathscr{L}$, we get the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{L} & \longrightarrow & \mathscr{K}(\mathscr{L}) & \longrightarrow & \mathscr{P}(\mathscr{L}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathscr{L} & \longrightarrow & \mathscr{L}(D) & \longrightarrow & \mathscr{L}(D)|_D & \longrightarrow & 0 \end{array} \qquad (\Phi_D)$$

Note that the sheaf $\mathscr{L}(D)|_D$ is supported at the support of D, and hence also has trivial cohomology. Taking global sections, we get:

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & \Gamma(C,\mathscr{L}) & \longrightarrow & \Gamma(C,\mathscr{K}(\mathscr{L})) & \longrightarrow & \Gamma(C,\mathscr{P}(\mathscr{L})) & \longrightarrow & H^1(C,\mathscr{L}) & \longrightarrow & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Gamma(C,\mathscr{L}) & \longrightarrow & \Gamma(C,\mathscr{L}(D)) & \longrightarrow & \Gamma(C,\mathscr{L}(D)|_D) & \xrightarrow{\delta_D} & H^1(C,\mathscr{L}) & \longrightarrow & H^1(C,\mathscr{L}(D)) & \longrightarrow & 0 \end{array} \qquad (\phi_D)$$

where the vertical maps are all induced by the above inclusions. Some key observations:

1. For increasing divisors D, the sequence below limits to the sequence above.
2. In particular, every class in $H^1(C,\mathscr{L})$ is in the image of δ_D for some D.
3. If $D = D_1 + D_2$, where D_1 and D_2 are effective divisors with disjoint supports, $\Gamma(C,\mathscr{L}(D)|_D) \cong \Gamma(C,\mathscr{L}(D_1)|_{D_1}) \oplus \Gamma(C,\mathscr{L}(D_2)|_{D_2})$, and hence δ_D can be identified with $\delta_{D_1} \oplus \delta_{D_2}$.

Theorem 1.2. *$H^1(C,\mathscr{L}) = 0$ iff for any effective divisor E and a point D of C, $\dim_k(\Gamma(C,\mathscr{L}(E+D))/\Gamma(C,\mathscr{L}(E))) = 1$.*

Proof. By observation 1, we see that $H^1(C,\mathscr{L})$ vanishes iff δ_D is zero for all D. Meanwhile, for a given D, $\delta_D = 0$ iff $\dim_k(\Gamma(C,\mathscr{L}(D))/\Gamma(C,\mathscr{L})) = \dim_k \Gamma(C,\mathscr{L}(D)|_D) = \deg(D)$. Therefore, we get $H^1(C,\mathscr{L}) = 0$ iff for all effective divisors D, $\dim_k(\Gamma(C,\mathscr{L}(D))/\Gamma(C,\mathscr{L})) = \deg(D)$. This is equivalent to the condition in the statement of the theorem by simple induction. ■

Using theorem 1.2, we can reduce the study of the sequence ϕ_D to the case where D is just a point, by replacing $\mathscr{L}$ by $\mathscr{L}(E)$.

2 Globalizing the sequence ϕ_D

We begin with the following sequence on $C \times C$:

$$0 \longrightarrow \mathscr{O}_{C\times C} \longrightarrow \mathscr{O}_{C\times C}(\Delta) \longrightarrow \mathscr{O}_{C\times C}(\Delta)|_\Delta \longrightarrow 0 \qquad (1)$$

We claim that this is the globalization of the sequence,

$$0 \longrightarrow \mathscr{O}_C \longrightarrow \mathscr{O}_C(D) \longrightarrow \mathscr{O}_C(D)|_D \longrightarrow 0 \qquad (2)$$

where D is a point of C. By that we mean that:

Theorem 2.1. *If $f_D : C \to C \times C$ is the map given by $\pi_1 \circ f_D = id_C$ and $\pi_2 \circ f_D$ is the constant map to the point D, the pullback of the sequence 1 by f_D gives us the sequence 2.*

Proof. On the open set $U = \Delta^c$, the sequence 1 is just,

$$0 \longrightarrow \mathscr{O}_U \longrightarrow \mathscr{O}_U \longrightarrow 0 \longrightarrow 0$$

And thus the pullback by f_D does in fact give the sequence (2) restricted to $f_D^{-1}(U) = C - D$. Thus we only need to check the pullback sequence on a neighbourhood of D.

Take an affine neighbourhood $W = \operatorname{Spec}(A)$ of D in C. Then, $V = W \times W = \operatorname{Spec}(B)$ is an affine neighbourhood of $f_D(D) = (D, D)$, where $B = A \otimes_k A$. Let $\mathfrak{p}_\Delta$ and $\mathfrak{p}_D$ be the prime ideals of B associated to the irreducible subsets $V \cap \Delta$ and $V \cap C \times D$, respectively. Then, $\mathfrak{p}_\Delta$ is the kernel of the diagonal map $B = A \otimes A \to A$, and $\mathfrak{p}_D$ is the kernel of the map given by $B = A \otimes A \to A \otimes A/m_D \cong A$, where m_D is the maximal ideal of A associated with the point D. Meanwhile, let $m_{(D,D)}$ be the maximal ideal of B associated to the point (D, D). We know that $\mathfrak{p}_\Delta + \mathfrak{p}_D \subseteq m_{(D,D)}$ Then we have the diagram,

$$\begin{array}{ccccc} B/\mathfrak{p}_\Delta & & & \searrow & \\ \uparrow & & & & \\ B & \longrightarrow & B/\mathfrak{p}_\Delta \otimes_B B/\mathfrak{p}_D & \xrightarrow{h} & B/(\mathfrak{p}_\Delta + \mathfrak{p}_D) \\ \downarrow & & & & \\ B/\mathfrak{p}_D & & & \nearrow & \end{array}$$

where the map h is the map induced by the universal property of the tensor product. Clearly, h is surjective. But $B/\mathfrak{p}_\Delta \otimes_B B/\mathfrak{p}_D \cong k$, therefore $B/(\mathfrak{p}_\Delta + \mathfrak{p}_D) \cong k$, and hence $\mathfrak{p}_\Delta + \mathfrak{p}_D = m_{(D,D)}$. This continues to hold after localizing at the $m_{(D,D)}$, and hence we get, $\mathfrak{p}_{(D,D),\Delta} + \mathfrak{p}_{(D,D),C\times D} = m_{C\times C,(D,D)}$, where $\mathfrak{p}_{(D,D),\Delta} = \mathfrak{p}_\Delta \mathscr{O}_{C\times C,(D,D)}$ and $\mathfrak{p}_{(D,D),C\times D} = \mathfrak{p}_D \mathscr{O}_{C\times C,(D,D)}$, are the prime ideals of the local ring $\mathscr{O}_{C\times C,(D,D)}$ of $C \times C$ at (D, D) corresponding to Δ and $C \times D$, respectively.
$C \times C$ is smooth, and hence $\mathscr{O}_{C\times C,(D,D)}$ is a UFD. In particular $\mathfrak{p}_{(D,D),\Delta} = (t_\Delta)$. Now, the map induced by f_D on the local rings, $f_D^* : \mathscr{O}_{C\times C,(D,D)} \to \mathscr{O}_{C,D}$, is just the quotient by $\mathfrak{p}_{(D,D),C\times D}$ map. Therefore, image of $\mathfrak{p}_{(D,D),\Delta}$ and in particular $t_D = f_D^*(t_\Delta)$ generate the maximal ideal $m_{C,D}$ of $\mathscr{O}_{C,D}$.
The rational function t_Δ is such that at all the divisors through (D, D), it has precisely just a simple zero across Δ. Removing the divisors other than Δ from the support of $\operatorname{div}(t_\Delta)$, we get an open set V_1 containing (D, D), such that $\operatorname{div}(f)|_{V_1} = \Delta$. Considering $t_D = f_D^*(t_\Delta)$ to be a regular function on $U_1 = f_D^{-1}(V_1)$, we find that t_D has a simple zero at D (since it generates $m_{C,D}$). Removing the points of the support of $\operatorname{div}(t_D)$ other than D from U_1, and their images from V_1, we get open neighbourhood V_2 of (D, D), such that $\operatorname{div}(t_\Delta)|_{V_2} = \Delta$, and $\operatorname{div}(t_D)|_{U_2} = D$, where $U_2 = f_D^{-1}(V_2)$, i.e. Δ and D are principal divisors on V_2 and U_2 respectively.
Coming back to our sequences, we see that when restricted to V_2, the sequence 1 becomes:

$$0 \longrightarrow \mathscr{O}_{V_2} \longrightarrow t_\Delta^{-1}\mathscr{O}_{V_2} \longrightarrow t_\Delta^{-1}\mathscr{O}_{V_2}/\mathscr{O}_{V_2} \longrightarrow 0$$

which when pulled back via f_D, becomes:

$$0 \longrightarrow \mathscr{O}_{U_2} \longrightarrow t_D^{-1}\mathscr{O}_{U_2} \longrightarrow t_D^{-1}\mathscr{O}_{U_2}/\mathscr{O}_{U_2} \longrightarrow 0$$

which is the same as the sequence 2 restricted to U_2. Since U and U_2 cover C, we are done. ∎

Tensoring the sequence 1 by $\pi_1^*\mathscr{L}$, we get:

$$0 \longrightarrow \pi_1^*\mathscr{L} \longrightarrow \pi_1^*\mathscr{L}(\Delta) \longrightarrow \pi_1^*\mathscr{L} \otimes \mathscr{O}_{C\times C}(\Delta)|_\Delta \longrightarrow 0$$

whose pullback via f_D is the sequence Φ_D of section 1. Using the adjunction of pullback and pushforward, we get the vertical maps of the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1^*\mathscr{L} & \longrightarrow & \pi_1^*\mathscr{L}(\Delta) & \longrightarrow & \pi_1^*\mathscr{L} \otimes \mathscr{O}_{C\times C}(\Delta)|_\Delta & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (f_D)_*\mathscr{L} & \longrightarrow & (f_D)_*\mathscr{L}(D) & \longrightarrow & (f_D)_*\ \mathscr{L}(D)|_D & \longrightarrow & 0 \end{array} \tag{3}$$

where the bottom row is exact since f_D is affine (it is a closed immersion), which makes $(f_D)_*$ exact. We now want to take the direct image of this diagram via π_2, but before that, a lemma for computation of higher direct images.

Lemma 2.2. *Let $f : X \to Y$ be a separated morphism of finite type of noetherian schemes, $u : Y' \to Y$ be a flat morphism of noetherian schemes and $g : X' = X \times_Y Y' \to Y'$ be the base extension of f via u. Let $\mathscr{F}$ be a quasi-coherent sheaf on X.*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ \downarrow g & & \downarrow f \\ Y' & \xrightarrow{u} & Y \end{array}$$

Then for $i \geq 0$ there are natural isomorphisms

$$u^* R^i f_*(\mathscr{F}) \cong R^i g_*(v^*\mathscr{F})$$

Proof. [Har77, Theorem III.9.3] ∎

Using the lemma for $f = u =$ the structure morphism $C \to \operatorname{Spec} k$, we get,

$$R^i \pi_{2*}(\pi_1 \mathscr{F}) \cong H^i(C, \mathscr{F}) \otimes_k \mathscr{O}_C$$

Also note that $\pi_2 \circ f_D$ is the constant map to the point D. Hence, $R^i(\pi_2 \circ f_D)_*(\mathscr{F})$ is the skyscraper sheaf associated to the module $H^i(C, \mathscr{F})$ supported at D. Also, we have natural isomorphisms:

$$\begin{aligned} \pi_{2*}(\pi_1^*\mathscr{L} \otimes \mathscr{O}_{C\times C}(\Delta)|_\Delta) &= \pi_{2*}(\pi_1^*\mathscr{L} \otimes \Delta_*\Delta^*(\mathscr{O}_{C\times C}(\Delta)|_\Delta)) \\ &= \pi_{2*}\Delta_*(\Delta^*(\pi_1^*\mathscr{L}) \otimes \Delta^*(\mathscr{O}_{C\times C}(\Delta)|_\Delta))) \\ &= \mathscr{L} \otimes \Delta^*(\mathscr{O}_{C\times C}(\Delta)|_\Delta) \\ &= \mathscr{L} \otimes \Omega_C^{\otimes -1} \end{aligned}$$

Finally we can take the pushforward of diagram 3 via π_2:

$$(\phi)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(C,\mathscr{L})\otimes_k \mathscr{O}_C & \longrightarrow & \pi_{2*}(\pi_1^*\mathscr{L}) & \xrightarrow{p} & \mathscr{L}\otimes\Omega_C^{\otimes -1} & \xrightarrow{\delta} & H^1(C,\mathscr{L})\otimes_k \mathscr{O}_C \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & i_{D*}\Gamma(C,\mathscr{L}) & \longrightarrow & i_{D*}\Gamma(C,\mathscr{L}(D)) & \xrightarrow{p_D} & i_{D*}\Gamma(C,\mathscr{L}(D)|_D) & \xrightarrow{\delta_D} & i_{D*}H^1(C,\mathscr{L}) \end{array}$$

Here, i_D is the inclusion of the point D into C. The sequence ϕ is the globalization of the sequence ϕ_D; Pulling back the diagram via i_D gives us the sequence ϕ_D in bottom rows, but all the vertical maps after pulling back are isomorphisms, so we get that $\phi_D = i_D^*(\phi)$. In particular, $\delta_D = 0$ iff p_D is surjective, which is true iff p is surjective on the stalk at D (by Nakayama's lemma) which is true iff δ is zero on the stalk at D. Therefore we have:

Theorem 2.3. *δ vanishes at D iff δ_D is 0.*

3 Finite dimensionality of the cohomology groups

Lemma 3.1. *For any non-zero effective divisor E on C, $H^1(C,\Omega_C(E)) = 0$.*

Proof. Set $\mathscr{L} = \Omega_C(E)$ for some non-zero effective divisor E in the sequences above. $\mathscr{L}\otimes\Omega_C^{\otimes -1} = \mathscr{O}_C(E)$. $H^1(C,\Omega_C(E))\otimes\mathscr{O}_C \cong \mathscr{O}_C^{\oplus r}$, where $r = \dim_k H^1(C,\Omega_C(E))$. If δ is nonzero, $r > 0$, and hence by composing δ with one of the projections, we will get a nonzero map $\mathscr{O}_C(E)\to\mathscr{O}_C$, which corresponds to a nonzero global section of $\mathscr{O}_C(-E)$, which do not exist (There is no non-zero regular function which vanishes only along E). Therefore δ must be zero, which means δ_D is zero for all D. But that gives us $\dim_k \Gamma(C,\Omega_C(E+D))/\Gamma(C,\Omega_C(E)) = 1$, for all non-zero effective divisors E, and all points D of C. Thus by theorem 1.2, we are done. ∎

Theorem 3.2. *Let $\mathscr{L}$ be an invertible sheaf on C. Then $\Gamma(C,\mathscr{L})$ and $H^1(C,\mathscr{L})$ are finite dimensional. Moreover, if we set the Euler characteristic of $\mathscr{L}$ to be $\chi(\mathscr{L}) = \dim_k\Gamma(C,\mathscr{L}) - \dim_k H^1(C,\mathscr{L})$, then*

$$\chi(\mathscr{L}) = 1 - g + \deg(\mathscr{L})$$

where $g = \dim_k H^1(C,\mathscr{O}_C)$ is the genus of C, and $\deg(\mathscr{L}) = \deg(div(s))$ for any rational section s of $\mathscr{L}$.

Proof. We know that for any rational section s of $\mathscr{L}$, the map $\mathscr{O}_C(D)\to\mathscr{L}$, given by $f\mapsto f*s$ is an isomorphism.
First we show the finite dimensionality of $\Gamma(C,\mathscr{L})$. If $\Gamma(C,\mathscr{L})\neq 0$, $\mathscr{L}$ has a global section s. $D = \mathrm{div}(s)$ will be an effective divisor, since s is globally defined, and hence $\mathscr{L}\cong\mathscr{O}_C(D)$ for an effective divisor D. But by the sequence ϕ_D, we know that $\dim_k\Gamma(C,\mathscr{O}_C(D))/\Gamma(C,\mathscr{O}_C)\le\deg(D)$. But $\dim_k\Gamma(C,\mathscr{O}_C) = 1$, which gives us the finite dimensionality of $\Gamma(C,\mathscr{O}_C(D)) = \Gamma(C,\mathscr{L})$. To see the finite dimensionality of

$H^1(C, \mathscr{L})$, observe that for any effective divisor E and invertible sheaf $\mathscr{L}$, the sequence ϕ_D tells us that $\mathscr{L}$ has finite dimensional cohomology iff $\mathscr{L}(E)$ does. Furthermore, if the cohomologies are finite dimensional, we have,

$$\dim_k \Gamma(C, \mathscr{L}(E)) - \dim_k \Gamma(C, \mathscr{L}) + \dim_k H^1(C, \mathscr{L}) - \dim_k H^1(C, \mathscr{L}(E)) \\ = \dim_k \Gamma(C, \mathscr{L}(E)|_E) = \deg(E)$$

that is, $\chi(\mathscr{L}(E)) = \chi(\mathscr{L}) + \deg(E)$. Writing any divisor D as $D_1 - D_2$, where D_1 and D_2 are effective, and using the result for effective divisors twice, we get the same result for all divisors.// By the above discussion and the previous lemma, we know that there exists an invertible sheaf $\mathscr{M}$ with finite dimensional cohomology groups. Now, $\mathscr{M} \cong \mathscr{O}(D_1)$ for a divisor D_1 of some rational section of $\mathscr{M}$. Thus $\mathscr{O}_C$ has finite dimensional cohomology groups. But as mentioned in the beginning of the proof, any invertible sheaf $\mathscr{L} \cong \mathscr{O}_C(D)$, which gives us that $\mathscr{L}$ has finite dimensional cohomologies and that $\chi(\mathscr{L}) = \chi(\mathscr{O}_C(D)) = \chi(\mathscr{O}_C) + \deg(D) = 1 - g + \deg(\mathscr{L})$. ■

4 The canonical class

First, a result on vanishing of cohomologies for sheafs with high degree.

Lemma 4.1. *Let $\mathscr{L}$ be an invertible sheaf on C with degree strictly larger than $\deg(\Omega_C)$. Then, $H^1(C, \mathscr{L}) = 0$.*

Proof. This is essentially a strengthening of lemma 3.1. The same proof goes through, except in this case there is no morphism from $\mathscr{N}(E) = \mathscr{L}(E) \otimes \Omega_C^{\otimes -1} \to \mathscr{O}_C$ for an effective divisor E, because $\mathscr{N}(E)$ has positive degree, hence a morphism $\mathscr{N} \to \mathscr{O}_C$ represents a regular section of $\mathscr{N}^{\otimes -1}$, which has negative degree. But an invertible sheaf of negative degree is isomorphic to $\mathscr{O}_C(D)$ for a divisor D of negative degree, which doesn't have any regular sections. ■

Theorem 4.2. *$H^1(C, \Omega_C)$ is one dimensional. In fact there exists a canonical isomorphism from $k \cong H^1(C, \Omega_C)$.*

Proof. By theorem 3.2, we know that for invertible sheaves $\mathscr{L}$ of low enough degree, $H^1(C, \mathscr{L})$ is non-trivial. Meanwhile, by the previous lemma, we know that $\mathscr{L}$ with sufficiently large degree has trivial cohomology.
Let $\mathscr{M}$ be an invertible sheaf of maximal possible degree with $H^1(C, \mathscr{M}) \neq 0$. Then for any point D, $\Gamma(C, M(D)) = 0$, that is for $\mathscr{L} = \mathscr{M}$, δ_D is surjective for all points D. This means that $0 < \dim_k H^1(C, \mathscr{M}) \leq \dim_k \Gamma(C, \mathscr{M}(D)|_D) = 1$, i.e. $\dim_k H^1(C, \mathscr{M}) = 1$. Since δ_D is surjective for all D, δ is surjective on the stalks at all points D, which means δ is an isomorphism, because the stalks at any point D of $\mathscr{M} \otimes \Omega_C^{\otimes -1}$ and $H^1(C, \mathscr{M}) \otimes_k \mathscr{O}_C \simeq \mathscr{O}_C$ are both free modules of rank 1 over the local ring at D, and any surjective map between such modules is an isomorphism. Thus, we get $\mathscr{M} \otimes \Omega_C^{\otimes -1} \simeq \mathscr{O}_C$, or equivalently, $\mathscr{M} \simeq \Omega_C$

To get the canonical map, observe that Ω_C itself satisfies the properties required of $\mathscr{M}$, in particular for $\mathscr{L} = \Omega_C$ as well, the map δ will be an isomorphism. But

$$\delta : \mathscr{O}_C \cong \Omega_C \otimes \Omega_C^{-1} \to H^1(C, \Omega_C)$$

Taking global sections, we get the canonical isomorphism. ■

5 Serre duality and the Riemann-Roch theorem

Given a section $s \in \Gamma(C, \Omega_C \otimes \mathscr{L}^{\otimes -1})$, we get a map $s \otimes 1 : \mathscr{L} \to \Omega_C \otimes \mathscr{L}^{\otimes -1} \otimes \mathscr{L} \cong \Omega_C$, which induces a map from $H^1(C, \mathscr{L}) \to H^1(C, \Omega_C)$. In particular we get a pairing

$$\Gamma(C, \Omega_C \otimes \mathscr{L}^{\otimes -1}) \times H^1(C, \mathscr{L}) \to H^1(C, \Omega_C) \cong k$$

which we denote as the cup product $\cup$.

Theorem 5.1. ***Serre duality***

$$\cup : \Gamma(C, \Omega_C \otimes \mathscr{L}^{\otimes -1}) \times H^1(C, \mathscr{L}) \to H^1(C, \Omega_C) \cong k$$

is a perfect pairing.

Proof. We are done if we show that the map $f : \Gamma(C, \Omega_C \otimes \mathscr{L}^{\otimes -1}) \to H^1(C, \mathscr{L})^\vee$, given by $s \mapsto c \circ (\cup s)$, is an isomorphism, where $c : H^1(C, \Omega_C) \to k$ is the canonical isomorphism.
Observe that the process of constructing the sequence ϕ and in particular the map δ is functorial in $\mathscr{L}$. Pick $s \in \Gamma(C, \Omega_C \otimes \mathscr{L}^{\otimes -1})$. We have the commutative diagram:

$$\begin{array}{lccc} \delta(\mathscr{L}) : & \mathscr{L} \otimes \Omega_C^{\otimes -1} & \longrightarrow & H^1(C, \mathscr{L}) \otimes_k \mathscr{O}_C \\ & \downarrow{\scriptstyle \times s} & & \downarrow{\scriptstyle \cup s \otimes 1} \\ \delta(\Omega_C) : & \mathscr{O}_C & \longrightarrow & H^1(C, \Omega_C) \otimes_k \mathscr{O}_C \end{array}$$

Define a map $\eta : H^1(C, \mathscr{L})^\vee \to \Gamma(C, \Omega_C \otimes \mathscr{L}^{\otimes -1})$, given by $t \mapsto (t \otimes 1) \circ \delta(\mathscr{L})$. Then from the diagram above, it can be seen that, $s = \delta(\Omega_C)^{-1} \circ (\cup s \otimes 1) \circ \delta(\mathscr{L}) = (c \otimes 1) \circ ((\cup s) \otimes 1) \circ \delta(\mathscr{L}) = ((c \circ (\cup s)) \otimes 1) \circ \delta(\mathscr{L}) = \eta \circ f(s)$. In particular, f is injective, η is surjective, and $\eta \circ f = \mathrm{id}$. If we show η is also injective, then $\eta = f^{-1}$ and we will be done.

Let $t \in \ker(\eta)$. Then $(t \otimes 1) \circ \delta(\mathscr{L}) = 0$. For each point D in C, pulling back via $i_D : \mathrm{Spec}(k(D)) \to C$ gives us $t \circ \delta_D = 0$. Pick a large n, specifically satisfying $n + \deg(\mathscr{L}) > \deg(\Omega_C)$. Pick n distinct points $D_1, D_2, \ldots, D_n$ on C. For each $1 \leq r \leq n$, $t \circ \delta_{D_r} = 0$, which means that $t \circ \delta_D = 0$ for $D = \sum_{r=1}^n D_r$. Since $\deg(\mathscr{L}(D)) = \deg(\mathscr{L}) + \deg(D) > \deg(\Omega_C)$, by lemma 4.1, $H^1(C, \mathscr{L}(D)) = 0$. Looking at sequence ϕ_D, this means that δ_D is surjective, and hence $t = 0$. This shows that η is injective and completes the proof. ■

Theorem 5.2. ***Riemann-Roch theorem***
For any invertible sheaf $\mathscr{L}$ *on* C*,*

$$\dim_k \Gamma(C, \mathscr{L}) - \dim_k \Gamma(C, \Omega_C \otimes \mathscr{L}^{\otimes -1}) = 1 - g + \deg(\mathscr{L})$$

Proof. By Serre duality, $\dim_k \Gamma(C, \Omega_C \otimes \mathscr{L}^{\otimes -1}) = \dim_k H^1(C, \mathscr{L})$. Therefore we have,

$$\begin{aligned} \dim_k \Gamma(C, \mathscr{L}) - \dim_k \Gamma(C, \Omega_C \otimes \mathscr{L}^{\otimes -1}) &= \dim_k \Gamma(C, \mathscr{L}) - \dim_k H^1(C, \mathscr{L}) \\ &= \chi(\mathscr{L}) \\ &= 1 - g + \deg(\mathscr{L}) \text{ by theorem 3.2} \end{aligned}$$

■

Corollary 5.2.1.

1. $\dim_k \Gamma(C, \Omega_C) = g$

2. $\deg(\Omega(C) = 2g - 2$

Proof.

1. Apply Riemann-Roch theorem for $\mathscr{L} = \mathscr{O}_C$. Since we know $\dim_k \Gamma(C, \mathscr{O}_C) = 1$, $\deg(\mathscr{O}_C) = 0$ and $\Omega_C \otimes \mathscr{O}_C^{\otimes -1} = \Omega_C$, we immediately get $\dim_k \Gamma(C, \Omega_C) = g$.

2. Apply Riemann-Roch theorem for $\mathscr{L} = \Omega_C$. Since we know $\dim_k \Gamma(C, \Omega_C) = g$, $\Omega_C \otimes \Omega_C^{\otimes -1} = \mathscr{O}_C$ and $\dim_k \Gamma(C, \mathscr{O}_C) = 1$, we immediately get $\deg(\Omega_C) = 2g - 2$.

■

References

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