The Smooth Representation associated to an Elliptic Curve

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Introduction

The Modularity theorem associates to a given elliptic curve defined over the field $\mathbb{Q}$ of rational numbers, a newform f such that the Tate module of the elliptic curve agrees with the Galois representation attached to f by Eichler, Shimura, Deligne and Serre. From the point of view of the Langlands program, one views newforms as automorphic representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. In this light, we give an explicit description of a local analogue of this; starting with an elliptic curve over a non-Archimedean local field F, we consider its Tate module and via the Local Langlands Correspondence for the group GL_2, attach to it a irreducible smooth representation of $\mathrm{GL}_2(F)$. We give an explicit description of the obtained representation in terms of the reduction type of the elliptic curve mod $\mathfrak{p}$, the maximal ideal of F, as mentioned in [Buz].

Chapter 1

Smooth Representations

In this chapter, we define the notion of smooth representations for locally profinite groups. One side of the Local Langlands Correspondence for GL_2 are smooth irreducible representations of $\mathrm{GL}_2(F)$ for a non-Archimedean local field F, so we discuss those in some detail.

1.1 Locally Profinite Groups

Definition. A **locally profinite group** is a topological group G such that its identity has a neighbourhood basis consisting of compact open subgroups.

Note that we assume all our topological groups to be Hausdorff. This is a fairly large class of groups, for instance it contains all discrete groups and profinite groups. In fact, compact locally profinite groups are exactly profinite groups, hence the name locally profinite. One can easily check that closed subgroups, quotients by closed normal subgroups and products of locally profinite groups are locally profinite. Now we go over some examples, establishing notation to be used throughout the rest of this document.

- The additive group of a non-Archimedean local field F.
  F is the field of fractions of a complete discrete valuation ring $\mathfrak{o}_F$ with a finite residue field k. Let $v_F : F^\times \to \mathbb{Z}$ be the valuation on F, and q be the cardinality of the residue field of characteristic p. The normalized absolute value is given by,

  $$||x|| = q^{-v_F(x)}, \quad x \in F^\times, \quad ||0|| = 0$$

  This induces a metric and topology on F. The fractional ideals

  $$\mathfrak{p}^n = \{x \in F \mid ||x|| \leq q^{-n}\}, \quad n \in \mathbb{Z}$$

  are open subgroups, topologically isomorphic to $\mathfrak{o}_F$ and hence compact. So we have,

  Proposition. *The additive group F is locally profinite, and F is the union of its compact open subgroups.*

- The multiplicative group $F^\times$ is also locally profinite. The subgroups $U_F = \mathfrak{o}_F^\times$ and $U_F^n = 1 + \mathfrak{p}^n, n \geq 1$, are compact open and gives a neighbourhood basis of 1 in $F^\times$.

- The vector space $F^n = F \times \cdots \times F$ is locally profinite, in particular the ring of $n \times n$ matrices $\mathrm{M}_n(F)$ is locally profinite under addition. Moreover $G = \mathrm{GL}_n(F)$ is an open subset of $\mathrm{M}_n(F)$, inversion of matrices is continuous, so G is a topological group. The subgroups

$$K = \mathrm{GL}_n(\mathfrak{o}_F), \quad K_j = 1 + \mathfrak{p}^j \mathrm{M}_n(\mathfrak{o}_F), j \geq 1,$$

are compact open, and give a neighbourhood basis of identity, so G is locally profinite.

1.2 Smooth Representations of Locally Profinite Groups

Throughout this section, G denotes a locally profinite group. We temporarily work over an arbitrary field C, however we will soon switch to $\mathbb{C}$.

Definition. Let (π, V) be a representation of G over C, that is, V is an C-vector space and $\pi : G \to \mathrm{Aut}_C(V)$ is a group homomorphism. We say (π, V) is **smooth**, if the action map $a_\pi : G \times V \to V$ given by $(g, v) \mapsto \pi(g)v$ is continuous with the discrete topology on V. We will often denote the representation (π, V) by just π.

For a representation (π, V) of G and a subset S of G, denote by V^S the subspace of $\pi(S)$-invariant elements of G, i.e.,

$$V^S := \{v \in V \mid \forall g \in S, \pi(g)v = v\}$$

Then we have the following equivalent condition for smoothness:

Lemma 1.2.1. *Let (π, V) be a representation of G over C. Then π is smooth iff every vector in V is fixed by a compact open subgroup, that is,*

$$V = \bigcup_K V^K$$

where the union is over compact open subgroups K of G.

Proof. Let π be smooth. For any vector $v \in V$, its stabilizer $G_v := \{g \in G \mid \pi(g)v = v\} = a_\pi^{-1}(\{v\}) \cap (G \times \{v\})$ is open by continuity of a_π. But since G is locally profinite, G_v contains a compact open subgroup K, or equivalently $v \in V^K$.
Let $V = \cup_K V^K$. It suffices to show that $a_\pi^{-1}(\{w\})$ is open for every $w \in V$. Let $(g, v) \in a_\pi^{-1}(\{w\})$, that is, $g \in G, v \in V$ such that $\pi(g)v = w$. By hypothesis, $v \in V^K$ for some compact open subgroup K of G. Then $gK \times \{v\}$ is an open neighbourhood of (g, v) contained in $a_\pi^{-1}(\{w\})$. Therefore $a_\pi^{-1}(\{w\})$ is open. □

We will essentially only use this equivalent condition when working with smooth representations.

Given a smooth representation (π, V), then any **G-subspace** (or **π-subspace** when π is not-obvious) W (i.e. $\pi(g)w \in W \ \forall g \in G, w \in W$) of V is also a smooth representation. Further, there is a natural representation of G on the quotient V/W, which is also smooth.

Definition. A smooth representation (π, V) is **irreducible** if $V \neq 0$ and V has no G-stable subspace other than V and 0.

Typically we'll deal with infinite dimensional representations. However, they might be satisfy a weak finiteness condition.

Definition. A smooth representation (π, V) is **admissible** if V^K is finite dimensional for each compact open subgroup K of G.

Definition. The **category** $\mathrm{Rep}_C(G)$ **of smooth representations of** G **over** C is the category consisting of the class of smooth representations of G over C as its objects, with morphisms given by **G-linear** maps, that is, $\mathrm{Hom}((\pi_1, V_1), (\pi_2, V_2))$ consists of linear maps $f : V_1 \to V_2$, satisfying

$$f \circ \pi_1(g) = \pi_2(g) \circ f, \quad g \in G$$

We will denote $\mathrm{Hom}((\pi_1, V_1), (\pi_2, V_2))$ by $\mathrm{Hom}(\pi_1, \pi_2)$ or $\mathrm{Hom}_G(\pi_1, \pi_2)$ when the underlying group is not obvious.
In chapter 2 we will also consider be the full subcategory $\mathrm{Rep}_C^f(G)$ on finite-dimensional smooth representations of G.

We point out that $\mathrm{Rep}(G)$ (resp. $\mathrm{Rep}^f(G)$) is an abelian category, since its a full subcategory of the category of all representations of G over C, and is closed under direct sums, kernels and cokernels.

We now take a look at what smooth representations look like in some simple cases.

Discrete groups are locally profinite, and for their representations, the smoothness condition is trivially satisfied. Something similar happens for more general locally profinite groups if the vector space is finite dimensional:

Proposition 1.2.2. *Let (π, V) be a finite dimensional representation of a locally profinite group G. Then π is smooth iff $\ker \pi$ is open, equivalently, π factors through a discrete group.*

Proof. Let π be smooth. Consider a basis $\{v_i\}_{i=1}^n$ of V. If v_i is fixed by the compact open subgroup K_i, then the compact open subgroup $K = \cap_i K_i$ fixes all of V and hence lies in the kernel. Therefore $\ker \pi$ is a open subgroup of G and the group $G/\ker \pi$ is discrete.
If $\ker \pi$ is open then the smoothness of π is trivial since every vector is fixed by $\ker \pi$ which must contain a compact open subgroup since G is locally profinite. □

We can do slightly better if the group is compact, i.e., profinite.

Proposition 1.2.3. *Let G be a profinite group.*

(i) Let (π, V) be a ***cyclic*** *smooth representation of a profinite group G, that is, V is the G-space generated by a single vector $v \in V$. Then V is finite dimensional.*

(ii) Let (τ, V) be a finite dimensional representation of G. Then τ is smooth iff $\ker \tau$ is open, equivalently, τ factors through a finite group.

Proof. Let $v \in V$ such that V is the G-space spanned by v. Since π is smooth, v is fixed by a compact open subgroup K. Since G is compact, $(G : K)$ is finite, hence $\{\pi(g)v \mid g \in G\} = \{\pi(g)v \mid g \in G/K\}$ is finite. This set spans V, so V is finite dimensional. Part (ii) is an immediate consequence of the previous proposition and the fact that compact discrete groups are finite. □

This means that cyclic (in particular irreducible) smooth representations of profinite groups are exactly cyclic (resp. irreducible) representations of its finite quotients.

At this point we will restrict ourselves to just the case of $L = \mathbb{C}$. We will also suppress the suppress the subscript and denote the category of complex smooth representations of G by just $\mathrm{Rep}(G)$.

Remark. Most of the general theory works for arbitrary uncountable and algebraically closed C of characteristic zero. The definition of smooth representation at no point refers to a topology on the base field, in particular if C is a field abstractly isomorphic to $\mathbb{C}$, then "changing scalars" gives an (additive) equivalence of categories between $\mathrm{Rep}_C(G)$ and $\mathrm{Rep}(G)$. In particular, any result about representations in $\mathrm{Rep}(G)$ gets carried over to $\mathrm{Rep}_C(G)$. In the next chapter, we will be using this for $C = \overline{\mathbb{Q}}_\ell$.

Proposition 1.2.4. *A representation* $(\chi, \mathbb{C})$ *is smooth iff* $\chi : G \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}) = \mathbb{C}^\times$ *is continuous.*

Proof. Let $(\chi, \mathbb{C})$ be a smooth representation of G. Since χ is smooth, 1 is fixed by a compact open subgroup K of G. The vector 1 spans $\mathbb{C}$, K fixes all of $\mathbb{C}$, so $\chi : G \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}) = \mathbb{C}^\times$ contains K in its kernel, in particular χ is a continuous homomorphism from G to $\mathbb{C}^\times$.
Now let χ be continuous. Take a small neighbourhood U of 1 in $\mathbb{C}^\times$ such that U does not contain any non-trivial subgroups of $\mathbb{C}^\times$. Then $\chi^{-1}(U)$ is a neighbourhood of identity in G, hence it contains a compact open subgroup H of G. But the image of H under χ is a subgroup of $\mathbb{C}^\times$ contained in U, so it must be the trivial subgroup. This means that the kernel of χ contains H, so the compact open subgroup H fixes all of $\mathbb{C}$ and hence χ is smooth. □

For an arbitrary topological group G, continous homomorphisms to $\mathbb{C}^\times$ are called **characters**, so the proposition can be stated as one-dimensional smooth representations of locally profinite groups are exactly their characters.

We introduce a size restriction on the group G to help us generalize familiar results to this setting:

Definition. We call a locally profinite group G **small**, if for any compact open subgroup K of G, the set G/K is countable.

Remark. It suffices to check the countability for just one compact open subgroup K. For any other compact open subgroup K', $K \cap K'$ is compact, open, and has finite index in K and K'. So $G/K \cap K'$ surjects onto G/K and G/K' with both surjections having finite fibres. As a consequence, G/K is countable iff G/K' is countable.
Note that this definition is not standard, and just made for making some statements convenient.

An easy consequence of this hypothesis is:

Lemma 1.2.5. *Let (π, V) be an irreducible smooth representation of a small group G. Then V has countable dimension.*

Proof. Let $v \in V$, $v \neq 0$. Then $v \in V^K$ for some compact open subgroup K of G. Since V is irreducible, it is spanned by $\{\pi(g)V \mid g \in G/K\}$ which is countable. □

Proposition 1.2.6 (Schur's Lemma)**.** *Let (π, V) be an irreducible smooth representation of a small group G, then* $\mathrm{End}_G(V) = \mathbb{C}$.

Proof. For any non-zero G-endomorphism ϕ, the image and kernel of ϕ are both G-subspaces of V. So ϕ is bijective and hence invertible. Therefore $\mathrm{End}_G(V)$ is a complex division algebra.
Fix $v \in V, v \neq 0$. The $\mathbb{C}$-linear map from $\mathrm{End}_G(V)$ to V given by $\phi \mapsto \phi(v)$ is injective, since if $\phi(v) = 0$, then its kernel is a non-zero G-subspace hence all of V. By the previous lemma, V has countable dimension, hence so does $\mathrm{End}_G(V)$. But if $\phi \in \mathrm{End}_G(V)$, $\phi \notin \mathbb{C}$, then it generates a transcendental extension $\mathbb{C}(\phi)$ over $\mathbb{C}$ inside $\mathrm{End}_G(V)$, which has uncountable dimension over $\mathbb{C}$. (For example, $\{(\phi - a)^{-1} \mid a \in \mathbb{C}\}$ is linearly independent) □

Corollary 1.2.6.1. *Let (π, V) be an irreducible smooth representation of a small group G. The centre Z of G, then acts on V via a character $\omega_\pi : Z \to \mathbb{C}^\times$, that is, $\pi(z)v - \omega_\pi(z)v$, for $v \in V$ and $z \in Z$.*

Proof. For any smooth representation (π, V) of G, there is a group homomorphism $Z \to \mathrm{End}_G(V)^\times$, given by $z \mapsto \phi_z$, where $\phi_z \in \mathrm{End}_G(V)$ given by $\phi_z(v) = \pi(z)v$. If V is irreducible, by Schur's Lemma, this becomes a homomorphism $\omega_\pi : Z \to \mathbb{C}^\times$. Moreover, if K is a compact open subgroup of G such that $V^K \neq 0$, for $z \in K \cap Z, v \in V^K$, $\omega_\pi(z)v = \phi_z(v) = \pi(z)v = v$, which implies $\omega_\pi(z) = 1$. Therefore the compact open subgroup $K \cap Z$ of Z lies in the kernel of ω_π, that is, ω_π is a character of Z. □

The character ω_π is called the **central character** of π.

Corollary 1.2.6.2. *If G is an abelian small group, then all its irreducible smooth representations are one-dimensional*

Proof. Follows immediately from the previous corollary. □

Characters of the additive group of a local field

We consider the additive group of a local field F to see some of these ideas in action. Let F be a local field. Then its ring of integers $\mathfrak{o}$ is a compact open subgroup, and $F/\mathfrak{o} = \cup_{n\geq 0}\mathfrak{p}^{-n}/\mathfrak{o}$. But $\mathfrak{p}^{-n}/\mathfrak{o}$ is finite for each n, so $F/\mathfrak{o}$ is countable, and hence F is small. Therefore irreducible smooth representations of F are exactly its characters. We give a complete description of the set of characters F. It forms a group under multiplication, which we denote by $\widehat{F}$.

Definition. Let $\psi \in \widehat{F}, \psi \neq 1$. Then the **level of** ψ is the least integer d such that $\mathfrak{p} \subset \ker\psi$.

Proposition 1.2.7 (Additive Duality)**.** *Let $\psi \in \widehat{F}, \psi \neq 1$, have level d.*

(i) Let $a \in F$. The map $a\psi : x \mapsto \psi(ax)$ is a character of F, and if $a \neq 0$, $a\psi$ has level $d - v_F(a)$.

(ii) The map $a \mapsto a\psi$ is a group isomorphism $F \cong \widehat{F}$.

Proof. Part (i) is an easy verification.
For part (ii), we have

$$(a+b)\psi(x) = \psi((a+b)x) = \psi(ax)\psi(bx) = (a\psi(x))(b\psi(x))$$

so $a \mapsto a\psi$ is a group homomorphism. Moreover, if $a\psi = 1$, then $\psi(ax) = 1$ for all $x \in F$, which is only possible if $a = 0$ since $\psi \neq 1$. Thus, $a \mapsto a\psi$ is injective.
To get surjectivity, let $\theta \in \widehat{F}, \theta \neq 1$ be of level l, and ϖ be a prime element of F. The character $\varpi^{d-l}\psi$ has level l, and hence agrees with θ on $\mathfrak{p}^l$. We now inductively construct elements $u_i \in U_F$, such that $u_i\varpi^{d-l}\psi$ agrees with θ on $\mathfrak{p}^{l-i}$ and $u_{i+1} \equiv u_i \pmod{\mathfrak{p}^i}$.
Start with $u_0 = 1$. Assume we have constructed $u_0, \ldots, u_i$ as required above. The character $\theta_i = \theta\cdot(u_i\varpi^{d-l}\psi)^{-1}$ is trivial on $\mathfrak{p}^{l-i}$. If θ_i is trivial on $\mathfrak{p}^{l-i-1}$, set $u_{i+1} = u_i$. If not, then consider the map

$$\begin{aligned} U_F &\to \widehat{\mathfrak{p}^{l-i-1}/\mathfrak{p}^{l-i}} \\ v &\mapsto v\varpi^{d-l+i}\psi \mid_{\mathfrak{p}^{l-i-1}} \end{aligned}$$

where $\widehat{\mathfrak{p}^{l-i-1}/\mathfrak{p}^{l-i}}$ denotes characters of $\mathfrak{p}^{l-i-1}$ which are trivial on $\mathfrak{p}^{l-i}$. For $v, v' \in U_F$, $v\varpi^{d-l+i}\psi \mid_{\mathfrak{p}^{l-i-1}} = v'\varpi^{d-l+i}\psi \mid_{\mathfrak{p}^{l-i-1}}$ iff $((v-v')\varpi^{d-l+i}\psi) \mid_{\mathfrak{p}^{l-i-1}}$ has level atmost $l-i-1$ iff $v \equiv v' \pmod{U_F^1}$. Therefore, the map above induces an injective map from U_F/U_F^1 to $\widehat{\mathfrak{p}^{l-i-1}/\mathfrak{p}^{l-i}}$. But the image of this map consists only of non-trivial characters because all of those characters have level $l-i$. Since $(U_F : U_F^1) = |\widehat{\mathfrak{p}^{l-i-1}/\mathfrak{p}^{l-i}} - \{1\}| = q-1$, the map is

surjective. Thus, there exists $v_i \in U_F$, $v_i\varpi^{d-l+i}\psi$ agrees with θ_i on $\mathfrak{p}^{l-i-1}$. Then $u_{i+1} = u_i + v_i\varpi^i$ satisfies the required conditions.
By completeness of F, there exists a $u \in U_F$ such that $u \equiv u_i \pmod{\mathfrak{p}^i}$. One can check that $\theta = u\varpi^{d-l}\psi$. □

1.2.1 Semisimplicity

Complex representations of finite groups are semisimple, i.e., they are a direct sum of irreducible representations. The same is not true for all smooth representations of locally profinite groups. We can however, recover some nice properties by restricting to compact open subgroup.

Proposition 1.2.8. *Let (π, V) be a smooth representation of a locally profinite group G. The following are equivalent:*

(i) V is a sum of its irreducible G-subspaces

(ii) V is the direct sum of a family of irreducible G-subspaces

(iii) any G-subspace of V has a G-complement in V.

Proof. The proof follows a standard argument used to show similar equivalences in many different contexts, for example, semisimplicity of modules over rings. For details, one can see Proposition 2.2 in [BH06]. □

Definition. Let H be a closed subgroup of a locally profinite group G. Then a smooth representation (π, V) of G is called **H-semisimple** if it satisfies the conditions of the previous proposition as a smooth representation of H. If $H = G$, we just say **semisimple** instead of G-semisimple.

Semisimplicity of a smooth representation can be checked on a finite index closed subgroup:

Lemma 1.2.9. *Let (π, V) be a smooth representation of G. If H is a finite index open subgroup of G and the π is H-semisimple, then it is G-semisimple.*

We will see later than the converse is also true, using the notion of induced representations.

Proof. Suppose U is a G-subspace of V. Then it is an H-subspace, and hence by H-semisimplicity, has an H-complement in V, say W. Let $f : V \to U$

be the projection with kernel W, this is an H-map. For a set of coset representatives $\{g_i\}_{i=1}^k$ for G/H, consider the map,

$$f^G : v \mapsto \frac{1}{(G:H)} \sum_{i=1}^{k} \pi(g_i) f(\pi(g_i^{-1})v), \quad v \in V$$

Since f is H-equivariant, the definition above does not depend on the choice of representatives g_i. In particular, for any $v \in V$ and $g \in G$,

$$\begin{aligned} f^G(\pi(g)v) &= \frac{1}{(G:H)} \sum_{i=1}^{k} \pi(g_i) f(\pi(g_i^{-1})\pi(g)v) \\ &= \pi(g) \left(\frac{1}{(G:H)} \sum_{i=1}^{k} \pi(g^{-1}g_i) f(\pi((g^{-1}g_i)^{-1})v) \right) \\ &= \pi(g) f^G(v) \end{aligned}$$

since $\{g^{-1}g_i\}_{i=1}^k$ form a set of coset representatives for G/H. Hence, f^G is a G-map. Moreover, it is easy to see that if $v \in U$, $f^G(v) = v$, i.e., f^G is a projection onto U. We conclude that $\ker f^G$ is a G-complement of U in V, and further that V is G-semisimple. □

Remark. A reader experienced in representation theory of finite groups might have noticed some similarity in proof above and the proof of Maschke's theorem on semisimplicity of complex representations of finite groups. In fact, applying the previous lemma with a finite (discrete) group for G and H as its trivial subgroup, one obtains Maschke's theorem on noting that for discrete groups, all representations are smooth and that all representations of the trivial group are semisimple.

Lemma 1.2.10. *Let G be a locally profinite group, and K a compact open subgroup of G. Then all smooth representations of G are K-semisimple. In particular, if G is profinite, all its smooth representations are semisimple.*

Proof. Let $(\pi.V)$ be a smooth representation of G. It suffices to show that every $v \in V$ is contained in a sum of irreducible K-subspaces of V. This follows from using proposition 1.2.3 on the K-subspace of V generated by v, seen as a cyclic smooth representation of the profinite group K, and Maschke's theorem (see remark above). □

Again, let G be a locally profinite group with a closed subgroup H. Let $\widehat{H}$ be the set of equivalence classes of irreducible smooth representations of H.

Definition. Let $\rho \in \widehat{H}$, and (π, V) be a smooth representation of G. The **ρ-isotypic component** of V, denoted by V^ρ, is defined to be the sum of all irreducible H-subspaces of V of class ρ.

Note that V^H is the isotypic component for the trivial representation of H.

Proposition 1.2.11. *Let (π, V) be a smooth representation of G which is H-semisimple for some closed subgroup H of G.*

(i) V is the direct sum of its H-isotypic components.

$$V = \bigoplus_{\rho \in \widehat{H}} V^\rho$$

(ii) Let (σ, W) be a smooth representation of G, and $f : V \to W$ be a G-homomorphism. Then for all $\rho \in \widehat{H}$,

$$f(V^\rho) \subset W^\rho \quad \textit{and} \quad W^\rho \cap f(V) = f(V^\rho)$$

Proof. Using lemma 1.2.10, V is a direct sum of a family $\{U_i\}_{i \in I}$ of irreducible H-subspaces. Let $U(\rho)$ be the sum of those U_i which are in the class ρ. Then we have

$$V = \bigoplus_{\rho \in \widehat{K}} U(\rho)$$

If W is an irreducible H-subspace of V of class ρ, then the inclusion $W \to V$ composed with the projection $V \to U_i$ is non-zero iff U_i is also in class ρ, otherwise there will be a non-zero homomorphism between two non-isomorphic irreducible representations of H. Therefore $U(\rho) = V^\rho$. Similar arguments can be used to show (ii). □

As corollaries we get,

Corollary 1.2.11.1. *A sequence*

$$U \xrightarrow{a} V \xrightarrow{b} W$$

of G-homomorphisms between smooth representations U, V and W of G is exact iff

$$U^K \xrightarrow{a} V^K \xrightarrow{b} W^K$$

is exact for every compact open subgroup K of G.

Proof. Follows from the fact that V is K-semisimple for all compact open subgroups K of G, along with part (ii) of the previous proposition and that for a smooth representation V, $V = \cup_K V^K$. □

Corollary 1.2.11.2. *Let (π, V) be a smooth representation of G, and K be a compact open subgroup of G. Let*

$$V(K) = \textit{linear span of } \{v - \pi(h)v \mid v \in V, h \in K\}$$

Then,

$$V(K) = \bigoplus_{\substack{\rho \in \widehat{K} \\ \rho \neq 1}} V^\rho, \quad V = V^K \oplus V(K),$$

and $V(K)$ is the unique K-complement of V^K in V.

Proof. Let $W = \bigoplus_{\rho \in \widehat{K}, \rho \neq 1} V^\rho$. Then by the previous proposition, W is K-complement of V^K. Therefore, there is a K-surjection $V \to V^K$ with kernel W. Since K acts trivially on V^K, $V(K)$ must lie in the kernel of this surjection. Meanwhile, if U is an irreducible K-subspace of V of class $\rho \neq 1$, then $U = U(K) \subset V(K)$. Therefore $V(K) = W$. □

1.2.2 Operations on Smooth representations

We consider smooth analogues of some basic operations on representations.

1.2.3 Induction

We discuss the notion of induced representations in this setting. Let G be a locally profinite group, and (σ, W) be a smooth representation of a closed subgroup H of G. Consider the space X of functions $f : G \to W$ which satisfy:

(i) $f(hg) = \sigma(h)f(g), h \in H, g \in G$

(ii) there exists a compact open subgroup K of G such that $f(gk) = f(g)$, for $g \in G, k \in K$.

We define a representation of G on W as follows. Let $\Sigma : G \to \mathrm{Aut}_{\mathbb{C}}(X)$ be given by,

$$\Sigma(g)f : x \mapsto f(xg)$$

Then (Σ, X) is a smooth representation of G, precisely because of the condition (ii) above.

Definition. The representation (Σ, X) constructed above from (σ, W) is called the representation of G **smoothly induced by** σ, and is denoted by $\operatorname{Ind}_H^G \sigma$.

The map $\sigma \mapsto \operatorname{Ind}_H^G \sigma$ gives a functor $\operatorname{Ind}_H^G : \operatorname{Rep}(H) \to \operatorname{Rep}(G)$. This comes with a natural H-homomorphism,

$$\begin{aligned} \alpha_\sigma : \operatorname{Ind}_H^G \sigma &\to W \\ f &\mapsto f(1) \end{aligned}$$

There is also a functor $\operatorname{Res}_H^G : \operatorname{Rep} G \to \operatorname{Rep} H$, which takes a smooth representation of G and views it as an H representation, called the **restriction** of the original representation to H. We will often write simply π for $\operatorname{Res}_H^G \pi$ where its clear what group we are considering π as a representation of. The functors Res_H^G and Ind_H^G form an adjoint pair:

Proposition 1.2.12 (Frobenius Reciprocity)**.** *Let H be a closed subgroup of a locally profinite group G. The smooth induction functor Ind_H^G is right-adjoint to the restriction functor Res_H^G. More precisely, given smooth representations (σ, W) of H and (π, V) of G, the map*

$$\begin{aligned} \operatorname{Hom}_G(\pi, \operatorname{Ind}_H^G \sigma) &\to \operatorname{Hom}_H(\pi, \sigma) \\ f &\mapsto \alpha_\sigma \circ f \end{aligned}$$

is an isomorphism which is functorial in both variables π, σ.

Proof. Let $t : V \to W$ be an H-homomorphism. Consider the G-homomorphism $t_* : V \to \operatorname{Ind} \sigma$ given by $t_*(v)(g) \mapsto t(\pi(g)v)$. Then $t \mapsto t_*$ is the inverse to the map above. We omit the proof of functoriality. □

We can consider a slight variation on above, where we consider the subspace X_c of X consisting of functions $f \in X$ which are compactly supported modulo H; that is

$$\operatorname{supp} f := \{g \in G \mid f(g) \neq 0\} \subset HC$$

for some compact subset C of G. The subspace X_c is G-stable.

Definition. The subrepresentation of $\operatorname{Ind}_H^G \sigma$ given by the subspace X_c is called the representation of G **compactly induced** or **smoothly induced with compact supports by** σ, and is denoted by $\text{c-}\operatorname{Ind}_H^G \sigma$.

As before, $\sigma \mapsto \text{c-Ind}_H^G \sigma$ gives a functor $\text{c-Ind}_H^G : \text{Rep}(H) \to \text{Rep}(G)$. By definition, the compact induction is defined as a subspace of the smooth induction, giving a natural transformation $\text{c-Ind}_H^G \to \text{Ind}_H^G$. Its easy to see that this is an isomorphism iff $H\backslash G$ is compact. Compact Induction satisfies a version of Frobenius Reciprocity with H is an open subgroup. In this case, there is natural H-homomorphism,

$$\begin{aligned} \alpha_\sigma^c : W &\to \text{c-Ind}_H^G \sigma \\ w &\mapsto f_w \end{aligned}$$

where $f_w(g) = \begin{cases} \sigma(g)w & g \in H \\ 0 & g \notin H \end{cases}$.

We'll use the following description of $\text{c-Ind}_H^G \sigma$:

Lemma 1.2.13. *Let (σ, W) be a smooth representation of an open subgroup H of G. Then,*

(i) The map α_σ^c is an H-isomorphism from W to the subspace of $\text{c-Ind}_H^G \sigma$ consisting of functions supported inside H.

(ii) Let $\mathcal{W}$ be a $\mathbb{C}$-basis of W and $\mathcal{G}$ a set of representatives for G/H. The set $\{gf_w \mid w \in \mathcal{W}, g \in \mathcal{G}\}$ is a $\mathbb{C}$-basis of $\text{c-Ind}\,\sigma$.

Proposition 1.2.14 (Frobenius Reciprocity)**.** *Let H be an open subgroup of a locally profinite group G. Then the compact induction functor c-Ind_H^G is left adjoint to the restriction functor Res_H^G. More precisely, given smooth representations (σ, W) of H and (π, V) of G, the map*

$$\begin{aligned} \text{Hom}_G(\text{c-Ind}_H^G \sigma, \pi) &\to \text{Hom}_H(\sigma, \pi) \\ f &\mapsto f \circ \alpha_\sigma^c \end{aligned}$$

is an isomorphism which is functorial in both variables.

Proof. Let t be an H-homomorphism $W \to V$. Set $t_* : \text{c-Ind}\,\sigma \to V$ as $t_*(gf_w) = \pi t(w), w \in W$. The map $t \mapsto t_*$ is the inverse of the map above. Again, we omit the proof of functoriality. □

Both the induction functors are quite well-behaved:

Proposition 1.2.15. *For any closed subgroup H of a locally profinite group G, the functors Ind_H^G and c-Ind_H^G are additive and exact.*

Under nice conditions, semisimplicity is preserved by induction. First, we need the promised converse to lemma 1.2.9:

Lemma 1.2.16. *Let H be a finite index open subgroup of G, and (σ, V) be a semisimple smooth representation of G. Then σ is H-semisimple.*

Proof. Since V can be written as a direct sum of irreducible G-subrepresentations, we reduce to the case where V is irreducible over G. Moreover, $H_0 = \bigcap_{g \in G/H} gHg^{-1} \subset H$ is an open normal subgroup of finite index in both H and G, so using lemma 1.2.9, we reduce to the case $H = H_0$ is normal in G.

Now, V is generated by a single non-zero vector over G, and hence by its translates by coset representatives of G/H over H; in particular it has a finite generating set over H. A standard Zorn's lemma argument implies the existence of a maximal H-subspace not containing all the generators, quotient by which gives a non-trivial irreducible H-quotient $(\sigma \mid_H, V) \to (\tau, U)$. By Frobenius Reciprocity, we get a G-map $V \to \operatorname{Ind}_H^G \tau$, which must be injective, since its non-trivial, so kernel isn't all of V, and V is irreducible over G. But $\operatorname{Ind}_H^G \tau = \text{c-}\operatorname{Ind}_H^G \tau = \oplus_{g \in G/H} \tau^g$ as an H-representation, where $\tau^g(x) = \tau(g^{-1}xg)$ is the conjugate of τ by g (see lemma 1.2.13(ii)). The representations τ^g are irreducible, so $\operatorname{Ind}_H^G \tau$ is H-semisimple. Hence its G-subspace V is also H-semisimple. □

Proposition 1.2.17. *Let H be a finite index open subgroup of G, and (σ, V) be a smooth representation of H. Then $\operatorname{Ind}_H^G \sigma$ is G-semisimple iff σ is H-semisimple.*

Proof. If $\operatorname{Ind}_H^G \sigma$ is G-semisimple, it is H-semisimple by the previous lemma. But then so is σ, since it appears as quotient via the map α_σ.

Now assume that σ is H-semisimple instead. Consider the finite index open normal subgroup $H_0 = \bigcap_{g \in G/H} gHg^{-1} \subset H$ as in the lemma above. Then starting with the identity map $\sigma \to \sigma$, functorially, we get the following maps,

$$\sigma \to \operatorname{Ind}_{H_0}^H \sigma$$
$$\operatorname{Ind}_H^G \sigma \to \operatorname{Ind}_H^G \operatorname{Ind}_{H_0}^H \sigma = \operatorname{Ind}_{H_0}^G \sigma$$

The first map is H-equivariant, and it can be easily seen to be injective by working out the definitions. The second map is obtained by applying Ind_H^G to the first, and hence is also injective. Now σ is H_0-semisimple (lemma 1.2.9), so by decomposing it into a direct sum of irreducibles over H_0 and then writing $\mathrm{Ind}_{H_0}^G \sigma$ as a sum of G-conjugates of these irreducibles, one sees that $\mathrm{Ind}_{H_0}^G \sigma$ is also H_0-semisimple. This implies that its G-subspace $\mathrm{Ind}_H^G \sigma$ is also H_0-semisimple, and hence G-semisimple by the previous lemma. □

1.2.4 Duality

We consider the notion of a dual representation of a smooth representation. Let (π, V) be a smooth representation of a locally profinite group G. Set $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$.
The space V^* comes with a representation π^* of G given by $\pi^*(g) = \pi(g^{-1})^*$, where $\pi(g^{-1})^*$ denotes the transpose of $\pi(g^{-1})$ given by $v^* \mapsto v^* \circ \pi(g^{-1})$ for all $g \in G, v^* \in V^*$. This representation is generally not smooth. Thus, as in the definition of smooth induction, we consider only those elements which are fixed by a compact open subgroup of G. We define,

$$\check{V} = \bigcup_K (V^*)^K$$

One can check that $\check{V}$ is a G-stable subspace of V. Denote the subrepresentation on $\check{V}$ by $(\check{\pi}, \check{V})$.

Definition. The smooth representation $(\check{\pi}, \check{V})$ is called the **contragredient** or **smooth dual** of (π, V).

We denote the evaluation map $\check{V} \times V \to \mathbb{C}$ as a pairing $\langle \check{v}, v \rangle = \check{v}(v)$. Then the representation $\check{\pi}$ satisfies

$$\langle \check{\pi}(g)\check{v}, v \rangle = \langle \check{v}, \pi(g^{-1})v \rangle, \quad g \in G, \check{v} \in \check{V}, v \in V$$

Proposition 1.2.18. *For any compact open subgroup K of G, restriction to V^K induces an isomorphism $\check{V}^K \cong (V^K)^*$.*

Proof. Let $\check{v} \in \check{V}^K$. For $v \in V, k \in K$,

$$\langle \check{v}, v - \pi(k)v \rangle = \langle \check{v}, v \rangle - \langle \check{\pi}(k^{-1})\check{v}, v \rangle = 0$$

since $\check{v} \in \check{V}$. Therefore $\langle \check{v}, V(K) \rangle = 0$. By corollary 1.2.11.2, $V = V^K \oplus V(K)$. Thus, $\check{v}$ is determined by its values on V^K, making the restriction map injective. Moreover, any linear functional on V^K can be extended to an element of $\check{V}^K$ by defining it to be zero on $V(K)$. □

Corollary 1.2.18.1. *Let (π, V) be a smooth representation of G, and $v \in V, v \neq 0$. Then there exists $\check{v} \in \check{V}$ such that $\langle \check{v}, v \rangle \neq 0$.*

Proof. Any nonzero $v \in V$ lies in V^K for some compact open subgroup K. The corollary follows from the previous proposition and the fact that there exists a functional f on V^K such that $f(v) \neq 0$. □

We consider now the smooth dual $(\check{\check{\pi}}, \check{\check{V}})$ of $(\check{\pi}, \check{V})$. There is a canonical G-map $\delta : V \to \check{\check{V}}$ given by,

$$\langle \delta(v), \check{v} \rangle_{\check{V}} = \langle \check{v}, v \rangle_V, \quad v \in V, \check{v} \in \check{V}.$$

where $\langle \cdot, \cdot \rangle_{\check{V}}$ and $\langle \cdot, \cdot \rangle_V$ denote the evaluation pairing for $\check{\check{V}}$ on $\check{V}$ and $\check{V}$ on V respectively. This map is injective by corollary 1.2.18.1.

Proposition 1.2.19. *Let (π, V) be a smooth representation of a locally profinite group G. The map $\delta : V \to \check{\check{V}}$ as defined above is an isomorphism iff π is admissible.*

Proof. By corollary 1.2.11.1, the map δ is an isomorphism iff the induced maps $\delta^K : V^K \to \check{\check{V}}^K$ are isomorphisms for all compact open subgroups K of G. But by proposition 1.2.18, δ^K is just the usual canonical map from the vector space V^K and its double dual $(V^K)^{**}$, which is an isomorphism iff V^K is finite dimensional. □

Now we construct the smooth dual functor. Let $(\pi, V), (\sigma, W)$ be smooth representations of G, and $f : V \to W$ be a G-map. We can define a map $\check{f} : \check{W} \to \check{V}$ as follows,

$$\langle \check{f}(\check{w}), v \rangle = \langle \check{w}, f(v) \rangle, \quad \check{w} \in \check{W}, v \in V$$

The map $\check{f}$ is a G-homomorphism, giving us a contravariant functor from $\mathrm{Rep}(G)$ to itself, given by $(\pi, V) \mapsto (\check{\pi}, \check{V})$.

Proposition 1.2.20. *The smooth dual functor is exact.*

Proof. Given an exact sequence of smooth representations (π_i, V_i) of G,

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

by corollary 1.2.11.1, the sequence

$$0 \to V_1^K \to V_2^K \to V_3^K \to 0$$

is exact for any compact open subgroup K of G. Taking duals is exact for vector spaces, so using proposition 1.2.18, we get that the sequence

$$0 \to \check{V}_3^K \to \check{V}_2^K \to \check{V}_1^K \to 0$$

is exact, which again from corollary 1.2.11.1 gives us that the sequence of smooth duals of V_i is exact. □

Proposition 1.2.21. *Let (π, V) be an admissible representation of G. Then (π, V) is irreducible iff $(\check{\pi}, \check{V})$ is irreducible.*

Proof. It follows from the previous proposition that if a representation is reducible, so is its dual. The result then follows from the fact that an admissible representation is isomorphic to its double dual (Proposition 1.2.19). □

1.2.5 Haar Measure

We go on a temporary digression about Haar measures on locally profinite groups which will be useful to us later. The almost discrete nature of the topology of profinite groups makes it so that their measure theory can be dealt with algebraically if we restrict ourselves to locally constant functions. We won't give proofs, but they can be found in section 3 of [BH06]. Let G be a locally profinite group. Denote by $C_c^\infty(G)$, the space of locally constant complex-valued functions on G with compact support. The group G acts by left translation λ and right translation ρ on $C_c^\infty(G)$:

$$\lambda(g)f : x \mapsto f(g^{-1}x)$$
$$\rho(g)f : x \mapsto f(xg)$$

for $g \in G$, $f \in C_c^\infty(G)$. The representations (C_c^∞, λ) and (C_c^∞, ρ) are smooth.

Definition. A **right Haar integral** on G is a non-zero linear functional $I : C_c^\infty(G) \to \mathbb{C}$ satisfying,

(i) $I(\rho(g)f) = I(f)$ for all $g \in G, f \in C_c^\infty(G)$, and

(ii) $I(f) \geq 0$ for any $f \in C_c^\infty(G)$ such that $f \geq 0$

A **left Haar integral** is defined similarly, with ρ replaced by λ in (i).

Proposition 1.2.22. *There exists a right Haar integral on any locally profinite group G. Moreover, if I and I' are two right Haar integrals, then $I' = cI$ for some $c > 0$.*

For any $f \in C_c^\infty(G)$, define $\check{(f)} \in C_c^\infty(G)$ as $\check{f}(g) = f(g^{-1})$. Then for any right Haar integral I, $I' : C_c^\infty(G) \to \mathbb{C}$ defined as, $I'(f) = I(\check{f})$, is a left Haar integral, since $I'(\lambda(g)f) = I(\rho(g)\check{f}) = I(\check{f}) = I'(f)$. Similarly, for any left Haar integral J, $J' : C_c^\infty(G) \to \mathbb{C}$ defined as, $J'(f) = I(\check{f})$ is a right Haar integral. So from the previous proposition, we get:

Proposition 1.2.23. *There exists a left Haar integral on any locally profinite group G. Moreover, if I and I' are two left Haar integrals, then $I' = cI$ for some $c > 0$.*

Let I be a left Haar integral on G. For any compact open subset S of G, let Γ_S be its characteristic function. Define,

$$\mu_I(S) = I(\Gamma_S)$$

Definition. The function μ_I on compact open subsets of G as defined above in terms of a left Haar integral I is called a **left Haar measure**. A **right Haar measure** can be defined similarly.

One can recover I from μ_I. Any $f \in C_c^\infty(G)$ can be written as $f = \sum_{i=1}^k a_i \Gamma_{S_i}$, where $a_i \in \mathbb{C}$, S_i are compact open sets. So $I(f) = \sum_{i=1}^k a_i \mu_I(S_i)$. The relationship between I and μ_I is denoted via the traditional notation:

$$I(f) = \int_G f(g) d\mu_I(g), \quad f \in C_c^\infty(G)$$

Lemma 1.2.24. *(i) μ_I is finitely additive, i.e., $\mu_I(S_1 \coprod S_2) = \mu_I(S_1) + \mu_I(S_2)$, for disjoint compact open subsets S_i of G.*

(ii) $\mu_I(gS) = \mu_I(S)$ for any $g \in G$, and compact open $S \subset G$.

(iii) $\mu_I(S) > 0$ for any non-empty compact open $S \subset G$.

Proof. Parts (i) and (ii) are clear from the definition. Any non-empty compact open set S is a finite union of left translates of compact open subgroups and given any two compact open subgroups H and H' of G, writing them both as a union of cosets of $H \cap H'$, we get $\mu_I(H) = \frac{(H:H\cap H')}{(H':H\cap H')}\mu_I(H')$. So if $\mu_I(H) = 0$ for any compact open subgroup H, then $\mu_I(H') = 0$ for all compact open subgroups H' and hence all compact open sets S, contradicting that $I \neq 0$. □

Let I be a left Haar integral on G. For a $g \in G$, consider $I_g : C_c^\infty(G) \to \mathbb{C}$ defined as, $I_g(f) = I(\rho(g)f)$. Then I_g is also a left Haar integral, since $I_g(\lambda(g')f) = I(\rho(g)(\lambda(g')f)) = I(\lambda(g')(\rho(g)f)) = I(\rho(g)f) = I_g(f)$ (left and right translation actions commute). By uniqueness of Haar integrals

$I_g = \delta_G(g)I$ for some $\delta_G(g) \in \mathbb{R}_+^\times$. Now pick another element $h \in G$. Then we have,

$$I(\rho(g)\rho(h)f) = I(\rho(gh)f) = I_{gh}(f) = \delta_G(gh)I(f)$$

$$I(\rho(g)\rho(h)f) = I_g(\rho(h)f) = \delta_G(g)I(\rho(h)f) = \delta_G(g)I_h(f) = \delta_G(g)\delta_G(h)I(f).$$

for any $f \in C_c^\infty(G)$. Therefore $\delta_G(gh) = \delta_G(g)\delta_G(h)$, and hence $\delta_G : G \to \mathbb{R}_+^\times$ is a group homomorphism. It is easy to see that δ_G does not depend on the choice of I.

Definition. The group homomorphism $\delta_G : G \to \mathbb{R}_+^\times$ defined as above is called the **module of** G.

Let K be a compact open subgroup of G, and $k \in K$. Then $\delta_G(k)\mu_I(K) = I_k(\Gamma_K) = I(\Gamma_K) = \mu_I(K)$. Since $\mu_I(K) > 0$ by the previous lemma, $\delta_G(k) = 1$, so δ_G is trivial on any compact open subgroup of G. In particular, δ_G is a character.

Definition. A locally profinite group G is called **unimodular** if its module δ_G is trivial.

One immediately concludes from the definitions,

Proposition 1.2.25. *A locally profinite group G is unimodular iff every left Haar integral is also a right Haar integral.*

We immediately conclude that abelian groups have trivial module. Moreover, since only compact subgroup of $\mathbb{R}_+^\times$ is trivial and the module is a character, compact groups are unimodular. In case the group in consideration is unimodular, we will drop the prefixes left and right from the Haar integrals and meassures.

Finally, we state without proof a couple results, one about invariant measures on quotient spaces, and another about the interaction of dual functors with the induction functors. Let G be a profinite group and H be a closed subgroup of G. Define

$$\delta_{H\backslash G} = \delta_H^{-1}\delta_G \mid_H : H \to \mathbb{R}_+^\times$$

Further, let $(\rho, C_c^\infty(H\backslash G, \delta_{H\backslash G})) = \text{c-Ind}_H^G \, \delta_{H\backslash G}$, i.e, the space of functions $f : G \to \mathbb{C}$ which are fixed by a compact open subgroup of G, are compactly supported modulo H, and satisfy $f(hg) = \delta_{H\backslash G}(h)f(g), h \in H, g \in G$, under the right translation action ρ.

Proposition 1.2.26. *There exists a non-zero functional* $I_H : C_c^\infty(H\backslash G, \delta_{H\backslash G}) \to \mathbb{C}$ *such that:*

(i) $I_H(\rho(g)f) = I_H(f)$, *for* $f \in C_c^\infty(H\backslash G, \delta_{H\backslash G}), g \in G$.

(ii) $I_H(f) \geq 0$, *for* $f \in C_c^\infty(H\backslash G, \delta_{H\backslash G}), f \geq 0$.

These conditions determine I_H *upto a positive constant factor.*

One uses notation:

$$I_H(f) = \int_{H\backslash G} f(g) d\mu_{H\backslash G}(g), \quad f \in C_c^\infty(H\backslash G, \delta_{H\backslash G})$$

and calls $\mu_{H\backslash G}$ a **positive semi-invariant measure** on $H\backslash G$.

Theorem 1.2.27 (Duality Theorem)**.** *Let* $\dot{\mu}$ *be a positive semi-invariant measure on* $H\backslash G$*, and* (σ, W) *be a smooth representation of* H*. There is a natural isomorphism,*

$$(\text{c-Ind}_H^G \sigma)^\vee \cong \text{Ind}_H^G(\delta_{H\backslash G} \otimes \check{\sigma})$$

depending only on the choice of $\dot{\mu}$*.*

1.3 Smooth Representations of $\mathrm{GL}_2(F)$

We finally discuss the case of $G = \mathrm{GL}_2(F)$, where F is a non-Archimedean local field.

1.3.1 Subgroups of $\mathrm{GL}_2(F)$

Let B, N and T denote the standard Borel subgroup, its unipotent radical and the standard maximal split torus in G, respectively. More explicitly:

$$B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in G \right\}, \quad N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in G \right\}$$
$$T = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \in G \right\}$$

We have a semi-direct product decomposition $B = T \ltimes N$. Also set $K_0 = \mathrm{GL}_2(\mathfrak{o}_F)$.
We collect here some facts about Haar measures on G and its subgroups.

Proposition 1.3.1. *Let* μ *be a Haar measure on* F*.*

(i) $\|x\|^{-1}d\mu$ *is a Haar measure on* $F^\times$*. More precisely, for* $f \in C_c^\infty(F^\times)$*, the function* $x \mapsto f(x)\|x\|^{-1}$ *(with the value 0 at* $x = 0$*) lies in* $C_c^\infty(F)$*, and*

$$f \mapsto \int_F f(x)\|x\|^{-1}d\mu(x)$$

is a Haar integral on $F^\times$*.*

(ii) $\mu(aS) = \|a\|\mu(S)$ *for every compact open subset* S *of* F *and* $a \in F^\times$*.*

(iii) For $a \in F^\times$*,*

$$\int_F f(ax)dx = \|a\|^{-1}\int_F f(x)dx$$

Let $A = \mathrm{M}_2(F)$ as an additive group. Then $A \cong F \times F \times F \times F$, hence all Haar measures on A can be obtained by taking a product of 4 copies of a Haar measure on F.

Proposition 1.3.2. *Let* μ^A *be a Haar measure on* A*. For* $f \in C_c^\infty(G)$*, the function* $x \mapsto f(x)\|\det x\|^{-2}$ *(with the value zero on* $A\backslash G$*) lies in* $C_c^\infty(A)$*. Moreover, the functional*

$$f \mapsto \int_A f(x)\|\det x\|^{-2}d\mu^A(x)$$

is a left and right Haar integral on G*. In particular,* G *is unimodular.*

Now we discuss the groups B, N and T. Since $N \cong F$ and $T \cong F^\times \times F^\times$, their Haar measures are easily obtained from Haar measures for F and $F^\times$.

Proposition 1.3.3. *The functional*

$$f \mapsto \int_T\int_N f(tn)d\mu_N(n)\mu_T(t), \quad f \in C_c^\infty(B)$$

is a left Haar integral on B*, where* μ_N *and* μ_T *are Haar integrals on* N *and* T *respectively. Moreover, the group* B *is not unimodular, with the module* δ_B *given by,*

$$\delta_B(tn) = \|t_2/t_1\|, \quad n \in N, t = \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right) \in T$$

We now discuss some useful decompositions of G and their consequences.

Theorem 1.3.4 (Iwasawa decomposition)**.** $G = BK_0$.

Proof. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$. If $c = 0$, $g \in B$. Otherwise, multiplying on the right by a permutation matrix in K_0 lets us assume $v(c) \geq v(d)$, where $v = v_F$ is the valuation on F. But then multiplying on the right by $\left(\begin{smallmatrix} 1 & 0 \\ -c/d & 1 \end{smallmatrix}\right) \in K_0$ we can again make the lower left entry zero. □

Since K_0 is compact, an immediate consequence of this is that $B\backslash G$ is compact. In particular,

Corollary 1.3.4.1. *For any smooth representation σ of B, the natural inclusion* $\text{c-Ind}_B^G \sigma \to \mathrm{Ind}_B^G \sigma$ *is an isomorphism.*

Since G is unimodular, δ_G is trivial. Hence $\delta_{B\backslash G} = \delta_B^{-1}$. Then from the duality theorem and the previous corollary,

Corollary 1.3.4.2 (Duality theorem)**.** *Let σ be a smooth representation of B. Fix a positive semi-invariant measure on $C_c^\infty(B\backslash G, \delta_B^{-1})$. There is a canonical isomorphism,*

$$(\mathrm{Ind}_B^G \sigma)^\vee \cong \mathrm{Ind}_B^G(\delta_B^{-1} \otimes \check{\sigma})$$

Theorem 1.3.5 (Cartan decompostion)**.** *Let ϖ be a prime element of F. The matrices*

$$\begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix}, a, b \in \mathbb{Z}, a \leq b$$

form a set of representatives for the double coset space $K_0\backslash G/K_0$.

Proof. We will only show that the matrices as above represent all the cosets, and not that they represent distinct cosets, since that is enough for our application. Given $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$, upto permutation matrices in K_0, we can assume that d has the largest absolute value. Then, by multiplying to the right and left by $\left(\begin{smallmatrix} 1 & 0 \\ -c/d & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & -b/d \\ 0 & 1 \end{smallmatrix}\right)$ respectively, we get a diagonal matrix in the same double coset. Then, multiplying by a diagonal matrix with unit entries, we get a matrix of the form as in the statement. □

Corollary 1.3.5.1. *G, B, N, T and K_0 are all small.*

Proof. From the Cartan decomposition it follows that the double coset decomposition $K_0\backslash G/K_0$ is countable. But any double coset $K_0 g K_0$ is a union of finitely many cosets $g' K_0$ since $K_0 g K_0$ is compact and K_0 is open. Thus G/K_0 is countable and G is small. Its easy to see that if H is a closed subgroup of G, then $H/H \cap K_0$ injects into G/K_0, showing that closed subgroups of small groups are small. □

1.3.2 Jacquet Module and the Principal Series

Let (π, V) be a smooth representation of G, and let $V(N)$ denote the subspace spanned by vectors of the form $v - \pi(n)v$, for $v \in V, n \in N$. Since N is normal in B, $V(N)$ is a B-subspace. Thus, $V_N := V/V(N)$ inherits a (smooth) representation π_N of $B/N = T$ (since N must act trivially).

Definition. The representation (π_N, V_N) is called the **Jacquet module** of (π, V) at N.

The map $(\pi, V) \mapsto (\pi_N, V_N)$ gives a functor from $\mathrm{Rep}(G)$ to $\mathrm{Rep}(T)$ called the *Jacquet functor*, which is exact and additive (see Lemma 8.1 in [BH06]).

Definition. An irreducible smooth representation (π, V) of G is said to be in the **principal series** or called a **principal series representation** if $\pi_N \neq 0$, otherwise it is said to be **cuspidal**.

Let (σ, W) be a smooth representation of T. We can view this as a representation of B which is trivial on N, and consider the representation $\mathrm{Ind}\,\sigma := \mathrm{Ind}_B^G\,\sigma$. If (π, V) is a smooth representation of G, by Frobenius Reciprocity, we have

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}\,\sigma) \cong \mathrm{Hom}_B(\pi, \sigma)$$

But N acts trivially on σ, so any B-map from V to W will factor through $V(N)$, so we have

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}\,\sigma) \cong \mathrm{Hom}_B(\pi, \sigma) \cong \mathrm{Hom}_T(\pi_N, \sigma)$$

We will use the above to get a useful description of the principal series representations.

Proposition 1.3.6. *Let (π, V) be an irreducible smooth representation of G. The following are equivalent:*

(i) π is in the principal series,

(ii) π is isomorphic to a G-subspace of a representation of the form $\mathrm{Ind}\,\chi$, for some character χ of T.

Proof. From the discussion above, if χ in any character of T,

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}\,\chi) \cong \mathrm{Hom}_T(\pi_N, \chi)$$

Since π is irreducible, any non-zero G-map from π to $\operatorname{Ind}\chi$ is going to give an isomorphism from π to a G-subspace of $\operatorname{Ind}\chi$. So (ii) is equivalent to the left hand side being non-zero for some χ.
Moreover, any non-zero map from π_N to χ is going to be a surjection, since χ is one dimensional. Meanwhile, any irreducible representation of T is a character, since T is (small) abelian. So RHS being non-zero is equivalent to π_N having an irreducible (T-)quotient, which is equivalent to (i) once we show that if $\pi_N \neq 0$ then it must have an irreducible quotient.
Pick $v \in V$, $v \neq 0$. Since V is irreducible over G, V is generated by just v over G. Now since π is smooth, v is fixed by a compact open subgroup K'. By the Iwasawa decomposition, $G = BK_0$, so V is generated over B by finitely many vectors $\{\pi(k)v \mid k \in K_0/K_0 \cap K'\} = \{v_1, \ldots, v_k\}$, and hence V_N is generated over T by their images $\{\bar{v}_1, \ldots, \bar{v}_k\}$
By an easy Zorn's lemma argument, there exists a T-subspace U of V_N which is maximal with the property $\bar{v}_i \notin U$. Then the quotient V_N/U is an irreducible T-quotient of V_N. □

An easy corollary is the admissibility of representations in the principal series:

Corollary 1.3.6.1.

(i) For any finite-dimensional representation (σ, W) *of* B*,* $\operatorname{Ind}_B^G \sigma$ *was admissible.*

(ii) All representations in the principal series are admissible.

Proof. Since characters of T are finite-dimensional representations of B, and subrepresentations of admissible representations are admissible, (ii) is an immediate consequence of (i) and the previous proposition.
For (i), let K be a compact open subgroup of G and $(\pi, V) = \operatorname{Ind}_B^G \sigma$. Then since $V^K \subset V^{K \cap K_0}$, it suffices to show V^K is finite-dimensional for compact open subgroups K contained in K_0. Since K and K_0 are compact open, $(K_0 : K)$ is finite. Iwasawa decomposition implies that $B\backslash G/K = B\backslash BK_0/K = \coprod_{i=1}^r Bg_iK$ is a finite double coset decomposition. Therefore the map $f \mapsto \{f(g_i)\}_{i=1}^k$ from V^K to $W^{\oplus k}$ is injective, and hence V^K has finite dimension. □

1.3.3 Classification of the Principal Series

We state the complete classification of the irreducible principal series representations. Proofs can be found in sections 9.6-9.11 of [BH06]. Given any character χ of T, then $\chi = \chi_1 \otimes \chi_2$ for characters χ_i of $F^\times$. More explicitly:

$$\chi : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \chi_1(a)\chi_2(b)$$

Then we have the following result:

Theorem 1.3.7 (Irreducibility Criterion). *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T. The representation $\mathrm{Ind}_B^G \chi$ is reducible iff $\chi_1\chi_2^{-1}$ is either the trivial character or the square of the norm character (i.e. $x \mapsto \|x\|^2$).*

We describe what happens in the reducible cases:

- If $\chi_1\chi_2^{-1}$ is trivial, then $\chi = \phi_T := \phi \circ \det$ for some character ϕ of $F^\times$. For $\phi = 1$, there exists an exact sequence,

  $$0 \to 1_G \to \mathrm{Ind}_B^G 1_T \to \mathrm{St}_G \to 0$$

  where the map $1_G \to \mathrm{Ind}_B^G$ is given by z going to the constant function with the value z and St_G is an irreducible representation of G called the **Steinberg representation**. For general ϕ,

  $$0 \to \phi_G \to \mathrm{Ind}_B^G \phi_T \to \phi_G \cdot \mathrm{St}_G \to 0$$

  where ϕ_T is ϕ_G restricted to T.

- If $\chi_1\chi_2^{-1}$ is square of the norm character. Then $\chi = \delta_B^{-1} \cdot \phi_G$ for some character ϕ of $F^\times$. Exactness of the smooth dual functor (1.2.20) and the duality theorem give the following exact sequence by taking duals of the sequence above,

  $$0 \to (\mathrm{St}_G)^\vee \to \mathrm{Ind}_B^G(\delta_B^{-1} \cdot \phi_T) \to \phi_G \to 0$$

  Using adjunction properties of the functor Ind_B^G, one can show $(\mathrm{St}_G)^\vee \cong \mathrm{St}_G$, so no new irreducible representation is obtained in this case.

We list out all the irreducible principal series representations. First, we introduce some notation which will be helpful later on. For any smooth representation σ of T, define

$$\iota_B^G \sigma = \mathrm{Ind}_B^G(\delta_B^{-1/2} \otimes \sigma)$$

Definition. The functor $\iota_B^G : \mathrm{Rep}(T) \to \mathrm{Rep}(G)$ is called **normalized** or **unitary smooth induction**.

If $\chi = \chi_1 \otimes \chi_2$ is a character of T, let χ^w denote the character $\chi_2 \otimes \chi_1$ of T.

Theorem 1.3.8 (Classification of Irreducible Principal Series Representations)**.** *The following is a complete list of isomorphism classes of irreducible representations of* $\mathrm{GL}_2(F)$ *in the principal series.*

(i) the induced representations $\iota_B^G \chi$*, where* $\chi \neq \phi_G \cdot \delta_B^{\pm\frac{1}{2}}$ *for any character* ϕ *of* $F^\times$*.*

(ii) the one-dimensional representations $\phi_G = \phi \circ \det$*, where* ϕ *ranges over the characters of* $F^\times$*.*

(iii) the representations $\phi \cdot \mathrm{St}_G$*, where* ϕ *ranges over the characters of* $F^\times$*.*

The classes in the list above have no overlaps, except that in (i), $\iota_B^G \chi \cong \iota_B^G \chi^w$*.*

Note that with this notation, the duality theorem says

$$(\iota_B^G \sigma)^\vee \cong \iota_B^G \check{\sigma}$$

For any smooth representation σ of B. Combining this with the self duality of the Steinberg representation, St_G, we can see that the dual of a principal series representation is also in the principal series. It follows from the admissibility of all smooth irreducible representations of G and propositions 1.2.19 and 1.2.21, that duals of cuspidal representations are also cuspidal.

1.3.4 L-functions and local constants

To any irreducible representation π of G, we attach a pair of invariants $L(\pi, s)$ and $\epsilon(\pi, s, \psi)$, following Godement and Jacquet, extending Tate's classical theory for GL_1. We will not give proofs for most of the results in this section. They can be found in chapter 6 of [BH06].

Throughout this section, fix a non-trivial character $\psi \in \widehat{F}$, and set $\psi_A = \psi \circ \mathrm{tr}_A$, where $\mathrm{tr}_A : A = \mathrm{M}_2(F) \to F$ denotes the trace function.

Definition. The **Fourier transform** $\hat{\Phi}$ of a function $\Phi \in C_c^\infty(A)$ **relative to a Haar measure** μ^A on A, is defined by the integral,

$$\hat{\Phi}(x) = \int_A \Phi(y)\psi_A(xy)d\mu^A(y)$$

Proposition 1.3.9.

(i) For $\Phi \in C_c^\infty(A)$, *the function* $\hat{\Phi}$ *also lies in* $C_c^\infty(A)$.

(ii) For a given ψ, *there is a unique Haar measure* μ_ψ^A *such that*

$$\hat{\hat{\Phi}}(x) = \Phi(-x) \quad , \Phi \in C_c^\infty(A), x \in A$$

(iii) Let $\mathfrak{M} = M_2(\mathfrak{o}_F)$. *The measure* μ_ψ^A *is determined by*

$$\mu_\psi^A(\mathfrak{M}) = q^{2l}$$

where l *is the level of* ψ.

Proof. Let $\Phi_{i,a} \in C_c^\infty(A)$ denote the characteristic function of $a + \mathfrak{p}^i\mathfrak{M}$ for $a \in A, i \in \mathbb{Z}$. These functions span $C_c^\infty(A)$, so it suffices to show (i) and (ii) for them. First we only consider $\Phi_i := \Phi_{i,0}$. As noted in section 1.3.1, Haar measures on A are given by product of Haar measures on four copies of F. Then it follows:

$$\begin{aligned}
\hat{\Phi}_i \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \int_A \Phi_i(y)\psi_A\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} y\right) d\mu(y) \\
&= \int_F\int_F\int_F\int_F \Phi_i\begin{pmatrix} p & q \\ r & s \end{pmatrix}\psi_A\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right) dpdqdrds \\
&= \int_{\mathfrak{p}^i}\int_{\mathfrak{p}^i}\int_{\mathfrak{p}^i}\int_{\mathfrak{p}^i} \psi(ap + br + cq + ds)dpdqdrds \\
&= \int_{\mathfrak{p}^i}\psi(ap)dp\int_{\mathfrak{p}^i}\psi(br)dr\int_{\mathfrak{p}^i}\psi(cq)dq\int_{\mathfrak{p}^i}\psi(ds)ds
\end{aligned}$$

where dp, dq, dr and ds are copies of a Haar measure μ on F. We compute one of the four terms, the computation for the rest follows similarly. If $a \in \mathfrak{p}^{l-i}$, then $ap \in \mathfrak{p}^l \subset \ker\psi$ for all $p \in \mathfrak{p}^i$, hence we get that the first term above is equal to $\mu(\mathfrak{p}^i)$.
If $a \notin \mathfrak{p}^{\mathrm{l-i}}$, then $a\mathfrak{p}^i \not\subset \ker\psi$, hence there exists $p_0 \in \mathfrak{p}^{\mathrm{i}}$ such that $\psi(ap_0) \neq 1$. But then by the translation invariance of dp,

$$\int_{\mathfrak{p}^i}\psi(ap)dp = \int_{\mathfrak{p}^i}\psi(a(p+p_0))dp = \psi(ap_0)\int_{\mathfrak{p}^i}\psi(ap)dp$$

so the integral is zero. Using properties of Haar measures on F (proposition 1.3.1). it follows that

$$\hat{\Phi}_i = \mu(\mathfrak{p}^i)^4\Phi_{l-i} = q^{-4i}\mu(\mathfrak{o}_F)^4\Phi_{l-i} = q^{-4i}\mu^A(\mathfrak{M})\Phi_{l-i} \in C_c^\infty(A)$$

Now we consider $\Phi_{i,a}$. Using the translation invariance of μ_A we get,

$$\begin{aligned}\hat{\Phi}_{i,a}(x) &= \int_A \Phi_{i,a}(y)\psi_A(xy)d\mu^A(y) \\ &= \int_A \Phi_i(y-a)\psi_A(xy)d\mu^A(y) \\ &= \int_A \Phi_i(y)\psi_A(x(y+a))d\mu^A(y) = \psi_A(xa)\hat{\Phi}_i \in C_c^\infty(A)\end{aligned}$$

for $x \in A$, which gives us (i). For (ii) and (iii), we compute $\hat{\hat{\Phi}}_{i,a}$:

$$\hat{\hat{\Phi}}_{i,a}(x) = \int_A \psi_A(ya)q^{-4i}\mu^A(\mathfrak{M})\Phi_{l-i}(y)\psi_A(xy)d\mu^A(y)$$

Since $\psi_A = \psi \circ \mathrm{tr}_A$, $\psi_A(ya) = \psi_A(ay)$. This gives,

$$\begin{aligned}\hat{\hat{\Phi}}_{i,a}(x) &= q^{-4i}\mu^A(\mathfrak{M})\int_A \Phi_{l-i}(y)\psi((x+a)y)d\mu^A(y) \\ &= q^{-4i}\mu^A(\mathfrak{M})\hat{\Phi}_{l-i}(x+a) \\ &= q^{-4l}\mu^A(\mathfrak{M})^2\Phi_i(x+a) \\ &= q^{-4l}\mu^A(\mathfrak{M})^2\Phi_{i,a}(-x)\end{aligned}$$

Parts (ii) and (iii) immediately follow. □

Definition. For a non-trivial character $\psi \in \widehat{F}$, the unique Haar measure μ_ψ^A satisfying

$$\hat{\hat{\Phi}}(x) = \Phi(-x) \quad x \in A, \Phi \in C_c^\infty(A)$$

as given by the last proposition is called the **self dual Haar measure on A, relative to** ψ.

We give a few more definitions. Let (π, V) be a smooth representation of G. For $v \in V, \check{v} \in \check{V}$, one can construct a function on G as,

$$\gamma_{\check{v}\otimes v} : g \mapsto \langle \check{v}, \pi(g)v \rangle$$

Definition. The vector space $\mathcal{C}(\pi)$ of functions spanned by the functions $\gamma_{\check{v}\otimes v}, \check{v} \in \check{V}, v \in V$ is called the space of **(matrix) coefficients of** π.

Now let (π, V) be an irreducible smooth representation of G and μ^* be a Haar measure on G. For $\Phi \in C_c^\infty(A)$ and $f \in \mathcal{C}(\pi)$, consider the integral,

$$\zeta(\Phi, f, s) = \int_G \Phi(x)f(x)\|\det x\|^s d\mu^*(x)$$

Theorem 1.3.10. *Let* (π, V) *be an irreducible representation of* G*.*

(i) There exists $s_0 \in \mathbb{R}$ *such that the integral defining* $\zeta(\Phi, f, s)$ *converges, absolutely and uniformly in vertical strips in the region* $\Re s > s_0$*, for all* Φ *and* f*. The integral represents a rational function in* q^{-s}*.*

(ii) Define

$$\mathcal{Z}(\pi) = \{\zeta(\Phi, f, s + \tfrac{1}{2}) : \Phi \in C_c^\infty, f \in \mathcal{C}(\pi)\}$$

There is a unique polynomial $P_\pi(X) \in \mathbb{C}[X]$*, satisfying* $P_\pi(0) = 1$*, and*

$$\mathcal{Z}(\pi) = P_\pi(q^{-s})^{-1}\mathbb{C}[q^s, q^{-s}]$$

.

Definition. The **L-function** $L(\pi, s)$ of an irreducible representation (π, V) of G is given by

$$L(\pi, s) = P_\pi(q^{-s})^{-1}$$

where $P_\pi(X) \in \mathbb{C}[X]$ is the polynomial given by the theorem above.

Next, we state the functional equation satisfied by the functions $\zeta(\Phi, f, s)$. For $f \in \mathcal{C}(\pi)$, denote by $\check{f}$ the function $g \mapsto f(g^{-1})$. Note that for $v \in V, \check{v} \in \check{V}$, $\check{\gamma}_{\check{v}\otimes v} = \gamma_{\delta(v)\otimes\check{v}} \in \mathcal{C}(\check{\pi})$, where $\delta : V \to \check{\check{V}}$ is the natural isomorphism between an admissible representation and its double dual (1.2.19). It follows that $f \mapsto \check{f}$ gives a linear isomorphism between $\mathcal{C}(\pi)$ and $\mathcal{C}(\check{\pi})$.

Theorem 1.3.11. *Let* (π, V) *be an irreducible smooth representation of* G*. There exists a unique rational function* $\gamma(\pi, s, \psi) \in \mathbb{C}(q^{-s})$ *such that*

$$\zeta(\hat{\Phi}, \check{f}, \tfrac{3}{2} - s) = \gamma(\pi, s, \psi)\zeta(\Phi, f, \tfrac{1}{2} + s) \tag{1.1}$$

for all $\Phi \in C_c^\infty(A), f \in \mathcal{C}(\pi)$*.*

Definition. The **local constant** $\varepsilon(\pi, s, \psi)$ of a smooth irreducible representation of G is a function defined by,

$$\varepsilon(\pi, s, \psi) = \gamma(\pi, s, \psi)\frac{L(\pi, s)}{L(\check{\pi}, 1 - s)}$$

Corollary 1.3.11.1. *The local constant* $\varepsilon(\pi, s, \psi)$ *satisfies the functional equation,*

$$\varepsilon(\pi, s, \psi)\varepsilon(\check{\pi}, 1 - s, \psi) = \omega_\pi(-1) \tag{1.2}$$

where ω_π *is the central character of* π*. Moreover, there exist* $a \in \mathbb{C}^\times$ *and* $b \in \mathbb{Z}$ *such that* $\varepsilon(\pi, s, \psi) = aq^{bs}$*.*

We need a couple lemmas.

Lemma 1.3.12. *For any Haar measure* μ^* *on* G*,*

$$\int_G \Phi(-x)d\mu^*(x) = \int_G \Phi(x)d\mu^*(x) \quad \Phi \in C_c^\infty(A)$$

Proof. The map $\Phi \mapsto \int_G \Phi(-x)d\mu^*(x)$ also defines a Haar integral. The result follows from using the uniqueness of Haar integrals and setting Φ to be the characteristic function of K_0. □

Lemma 1.3.13. *For* $f \in \mathcal{C}(\pi), z \in Z, g \in G,$ $f(zg) = \omega_\pi(z)f(g)$*.*

Proof. Since $\mathcal{C}(\pi)$ is spanned by $\gamma_{\check{v}\otimes v}, \check{v} \in \check{V}, v \in V$, it suffices to prove the statement for $f = \gamma_{\check{v}\otimes v}$. But then,

$$\gamma_{\check{v}\otimes v}(zg) = \langle \check{v}, \pi(zg)v\rangle = \omega_\pi(z)\langle \check{v}, \pi(g)v\rangle = \omega_\pi(z)\gamma_{\check{v}\otimes v}(g)$$

□

Proof of Corollary 1.2. Using the functional equation 1.1 twice, once for $(\hat{\Phi}, \check{f})$ and for (Φ, f),

$$\begin{aligned}\zeta(\hat{\hat{\Phi}}, f, \tfrac{1}{2} + s) &= \gamma(\check{\pi}, 1-s, \psi)\zeta(\hat{\Phi}, \check{f}, \tfrac{3}{2} - s) \\ &= \gamma(\check{\pi}, 1-s, \psi)\gamma(\pi, s, \psi)\zeta(\Phi, f, \tfrac{1}{2} + s) \\ &= \varepsilon(\check{\pi}, 1-s, \psi)\varepsilon(\pi, s, \psi)\zeta(\Phi, f, \tfrac{1}{2} + s)\end{aligned}$$

Using Fourier inversion and the lemmas above,

$$\begin{aligned}\zeta(\hat{\hat{\Phi}}, f, s) &= \int_G \Phi(-x)f(x)\|\det x\|^s d\mu^*(x) \\ &= \int_G \Phi(x)f(-x)\|\det(-x)\|^s d\mu^*(x) && \text{(by lemma 1.3.12)} \\ &= \omega_\pi(-1)\int_G \Phi(x)f(x)\|\det x\|^s d\mu^*(x) && \text{(by lemma 1.3.13)} \\ &= \omega_\pi(-1)\zeta(\Phi, f, s)\end{aligned}$$

Plugging this into the functional equation obtained above, (1.2) follows. To show that $\varepsilon(\pi, s, \psi)$ is of the required form, first note that we have

$$L(\pi, s) = \sum_{i=1}^r \zeta(\Phi_i, f_i, s + \tfrac{1}{2})$$

for some $\Phi_i \in C_c^\infty(A), f_i \in \mathcal{C}(\pi)$ by definition of $L(\pi, s)$. The definition of $\varepsilon(\pi, s, \psi)$ combined with the functional equation 1.1 then gives,

$$\varepsilon(\pi, s, \psi) = L(\check{\pi}, 1-s)^{-1} \sum_{i=1}^{r} \zeta(\hat{\Phi}_i, \check{f}_i, \tfrac{3}{2} - s)$$

The right hand side lies in $\mathbb{C}[q^s, q^{-s}]$ by the defining property of the L-function $L(\check{\pi}, s)$. Similarly, $\varepsilon(\check{\pi}, 1-s, \psi)$ lies in $\mathbb{C}[q^s, q^{-s}]$. But then the functional equation 1.2 implies that $\varepsilon(\pi, s, \psi)$ is a unit in $\mathbb{C}[q^s, q^{-s}]$, and hence it must be of the required form. □

Recall Additive Duality: $a \mapsto a\psi$ is an isomorphism $F \cong \widehat{F}$. The L-function does not depend on the chosen character ψ, meanwhile the local constant is effected in a very controlled manner:

Proposition 1.3.14. *Let* (π, V) *be an irreducible smooth representation of* G*, and* $a \in F^\times$*. Then,*

$$\varepsilon(\pi, s, a\psi) = \omega_\pi(a)\|a\|^{2s-1}\varepsilon(\pi, s, \psi)$$

L-functions and local constants form essentially complete invariants for irreducible smooth representations of G.

Theorem 1.3.15 (Converse Theorem)**.** *Let* π_1, π_2 *be irreducible smooth representations of* G*. Suppose that*

$$L(\chi\pi_1, s) = L(\chi\pi_2, s) \quad \varepsilon(\chi\pi_1, s, \psi) = \varepsilon(\chi\pi_2, s, \psi),$$

for all characters χ *of* $F^\times$*. Then we have* $\pi_1 \cong \pi_2$*.*

We will later give a proof of a slight strengthening of the converse theorem for the principal series representations, after stating theit L-functions and local constants. To state these invariants for representations of GL_2, we first need the same for the case of GL_1.

L-functions and local constants for GL_1

Since $\mathrm{GL}_1(F) = F^\times$ is a small locally profinite group, its irreducible smooth representations are exactly its characters. An account of the GL_1 theory in the notation of this document can be found in section 23 of [BH06]. First a few definitions,

Definition. The **level** of character χ of $F^\times$ is defined to be the least integer $n \geq 0$ such that $U_F^{n+1} \subset \ker\chi$.

Definition. A character χ is said to be **unramified**, if its trivial on U_F.

The L-functions are easy to describe:

Proposition 1.3.16. *Let χ be a character of $F^\times$ and ϖ be a prime element of F. Then,*

$$L(\chi, s) = \begin{cases} (1-\chi(\varpi)q^{-s})^{-1} & \text{if } \chi \text{ is unramified,} \\ 1 & \text{otherwise.} \end{cases}$$

Now, for the local constants,

Proposition 1.3.17. *Let χ be an unramified character of $F^\times$, ψ be of level one and ϖ be a prime element of F. Then,*

$$\varepsilon(\chi, s, \psi) = q^{s-\frac{1}{2}}\chi(\varpi)^{-1}$$

Proposition 1.3.18. *Let χ be a character of level $n \geq 0$ which is not unramified, and ψ be of level one. Then,*

$$\varepsilon(\chi, s, \psi) = q^{n\left(\frac{1}{2}-s\right)} \sum_{x \in U_F/U_F^{n+1}} \chi(\alpha x)^{-1}\psi(\alpha x)/q^{(n+1)/2}$$

for any $\alpha \in F^\times$ such that $v_F(\alpha) = -n$.

L-functions and local constants GL_2

Theorem 1.3.19. *Let $\chi = \chi_1 \otimes \chi_2$ be a character of the group T, and let π be an irreducible principal series representation which is a G-composition factor of $\iota_B^G\chi$. For any $\psi \in \widehat{F}, \psi \neq 1$, we have*

$$L(\pi, s) = L(\chi_1, s)L(\chi_2, s)$$
$$\varepsilon(\pi, s, \psi) = \varepsilon(\chi_1, s, \psi)\varepsilon(\chi_2, s, \psi)$$

except when $\pi \cong \chi \cdot \mathrm{St}_G$, for an unramified character χ of $F^\times$. In this exceptional case,

$$L(\pi, s) = L(\chi, s + \tfrac{1}{2}), \quad \varepsilon(\pi, s, \psi) = -\varepsilon(\chi, s, \psi)$$

Proposition 1.3.20. *(i) Let (π, V) be an irreducible principal series representation of $\mathrm{GL}_2(F)$. Then there exists a character χ of $F^\times$ such that the L-function $L(\chi\pi, s)$ is not constant.*

(ii) *(Converse Theorem for the Principal Series) Let* $\pi_i, i = 1, 2$ *be irreducible principal series representations of* $\mathrm{GL}_2(F)$*. Then if*

$$L(\chi\pi_1, s) = L(\chi\pi_2, s)$$

for all characters χ *of* $F^\times$*, then* $\pi_1 \cong \pi_2$*.*

Proof. (i) Let π be a composition factor of $\iota_B^G\chi$ for a character $\chi = \chi_1 \otimes \chi_2$ of T. Then $\chi_1^{-1}\pi$ is a composition factor of $\iota_B^G(1 \otimes \chi_1^{-1}\chi_2)$, and thus $L(\chi_1^{-1} \cdot \pi, s)$ is cannot be a constant function.

(ii) We prove that an irreducible principal series representation π is determined by the map $\chi \mapsto L(\chi\pi, s)$ using the description of the L-functions in theorem 1.3.19. Suppose there exists a character χ of $F^\times$ such that the $L(\chi\pi, s)$ has degree 2. Then $L(\chi\pi, s) = L(\chi_1, s)L(\chi_2, s)$ for some unramified characters χ_1, χ_2 of $F^\times$.
If $\chi_1\chi_2^{-1} \neq \|\cdot\|^{\pm 1}$, then $\iota_B^G(\chi_1 \otimes \chi_2)$ is irreducible by theorem 1.3.8, therefore $\pi \cong \chi^{-1}\iota_B^G(\chi_1 \otimes \chi_2) = \iota_B^G(\chi^{-1}\chi_1 \otimes \chi^{-1}\chi_2)$.
Otherwise $\{\chi_1, \chi_2\} = \{\phi\|\cdot\|^{\frac{1}{2}}, \phi\|\cdot\|^{-\frac{1}{2}}\}$ for some unramified character ϕ of $F^\times$, in which case $\pi = \chi^{-1}\phi \circ \det$.
Now suppose $L(\chi\pi, s)$ has degree atmost 1 for all characters χ. By (i), there exists a χ such that $L(\chi\pi, s)$ has degree 1. Then $\chi\pi$ is of the form $\iota_B^G(\theta' \otimes \theta)$ for an unramified character θ and a ramified character θ' or $\chi\pi$ is of the form $\theta\,\mathrm{St}_G$ for an unramified character θ. The cases can be distinguished by the existence of an ramified character ϕ such that $L(\phi\chi\pi, s)$ is not constant, since in the first case, we can take $\phi = \theta'^{-1}$ and in the second case, no such ϕ exists. In the latter case, $\pi \cong \chi^{-1}\theta\,\mathrm{St}_G$, where θ is determined by $L(\chi\pi, s) = L(\theta, s + \frac{1}{2})$.
In the former case, suppose $L(\chi\pi, s) = L(\theta, s)$ and $L(\phi\chi\pi, s) = L(\theta'', s)$ for unramified characters θ and θ'' and a ramified character θ. Then we must have $\pi \cong \chi^{-1}\iota_B^G(\theta''\phi^{-1} \otimes \theta)$. □

Theorem 1.3.21. *Let* (π, V) *be an irreducible cuspidal representation of* $\mathrm{GL}_2(F)$*. Then its L-function is trivial, that is,*

$$L(\pi, s) = 1$$

Chapter 2

Weil-Deligne Representations

We now take a look at the arithmetic side of Local Langlands, starting with a look into the structure of the absolute Galois group of a non-Archimedean local field.

2.1 Absolute Galois group of a local field

Fix a separable closure $\bar{F}$ of a non-Archimedean local field F, and let $\Omega_F = \mathrm{Gal}(\bar{F}/F)$. We recall some facts from Galois theory. Galois groups are profinite with a natural (Krull) topology, given by

$$\Omega_F = \varprojlim \mathrm{Gal}(E/F)$$

as E/F ranges over finite Galois extensions contained in $\bar{F}$. We fix the convention that any field extension of F is contained inside $\bar{F}$. Such field extensions K/F are in natural bijection with closed subgroups of Ω_F which can be identified with Ω_E. Moreover, K/F is finite iff Ω_E is open, and in this case $(\Omega_F : \Omega_E) = [K : F]$.
We collect here some facts about extensions of F. There is tower of extensions;

E	$\mathrm{Gal}(\bar{F}/E)$
$\bar{F}$	e
$\vert$	$\cap$
F^{tr}	$\mathcal{P}_F$
$\vert$	$\cap$
F^{ur}	$\mathcal{I}_F$
$\vert$	$\cap$
F	Ω_F

Here F^{ur} and F^{tr} denote the **maximal unramified** and **maximal tamely ramified extensions** of F respectively. Their corresponding subgroups $\mathcal{I}_F$

and $\mathcal{P}_F$ of Ω_F are called the **inertia** and the **wild inertia groups** of F. We describe them in more detail.
Any F-automorphism of $\bar{F}$ induces a k_F-automorphism of $k_{\bar{F}}$, where k_E denotes the residue field of an extension E of F. The residue field $k_{\bar{F}}$ is an algebraic closure of k_F. Since k_F is a finite field of order q, this induces a map,

$$V_F : \Omega_F \to \mathrm{Gal}(k_{\bar{F}}/k_F) \cong \widehat{\mathbb{Z}} \tag{2.1}$$

where the last isomorphism given by sending the inverse of the Frobenius automorphism ($x \mapsto x^q$) to 1, and $\widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$ denotes the profinite completion of $\mathbb{Z}$. This map is surjective, and the kernel of this map is the inertia group $\mathcal{I}_F$. The corresponding extension F^{ur} is generated by n^{th} roots of unity for all $n \in \mathbb{N}$ prime to $p := \mathrm{char}(k_F)$. The map V_F induces a short exact sequence of topological groups,

$$1 \longrightarrow \mathcal{I}_F \longrightarrow \Omega_F \xrightarrow{V_F} \widehat{\mathbb{Z}} \longrightarrow 1 \tag{2.2}$$

Definition. The element $\Phi_F \in \mathrm{Gal}(F^{ur}/F)$ which acts as the inverse of the Frobenius on the residue field is called the **geometric Frobenius substitution** on F^{ur}. It is the unique element of $\mathrm{Gal}(F^{ur}/F) = \Omega_F/\mathcal{I}_F$ satisfying $V_F(\Phi_F) = 1$.
A lift of Φ_F to Ω_F is called a **(geometric) Frobenius element (over F)**.

Next, for each integer $n \geq 1$, $p \nmid n$, there is a unique extension E^n/F^{ur} of degree n. If ϖ is a prime element of F, E_n is generated by an n^{th} root of ϖ. The extension F^{tr} is the composite of the extensions E_n. Note that F^{tr} is normal over F as well, since it is generated by the splitting fields of $x^n - \varpi$ for all n prime to p.
We describe the Galois group $\mathrm{Gal}(F^{tr}/F^{ur}) = \mathcal{I}_F/\mathcal{P}_F$. Let $\alpha \in E_n$ such that $\alpha^n = \varpi$. Then there is an isomorphism,

$$\begin{aligned} T_n : \mathrm{Gal}(K_n/F^{ur}) &\to \boldsymbol{\mu}_n \\ \sigma &\mapsto \sigma(\alpha)/\alpha, \end{aligned} \tag{2.3}$$

where $\boldsymbol{\mu}_n$ is the group of n^{th}-roots of unity. This isomorphism does not depend on the choice of α. Taking inverse limits we get a topological isomorphism,

$$\mathcal{I}_F/\mathcal{P}_F \cong \varprojlim_{p \nmid n} \boldsymbol{\mu}_n \cong \prod_{\ell \neq p} \mathbb{Z}_\ell \tag{2.4}$$

where the last isomorphism is given by the Chinese Remainder Theorem.
The extension F^{ur}/F is abelian, however F^{tr}/F is not. The Galois group

$\mathrm{Gal}(F^{tr}/F) = \Omega_F/\mathcal{P}_F$ acts by conjugation on $\mathcal{I}_F/\mathcal{P}_F$. But $\mathcal{I}_F/\mathcal{P}_F$ is abelian, so the action factors through $\Omega_F/\mathcal{I}_F$. The maps in 2.4 give a topological isomorphism between $\mathcal{I}_F/\mathcal{P}_F$ and $\prod_{\ell \neq p} \mathbb{Z}_\ell$. We can use this isomorphism to give a more explicit description of the conjugation action.

Proposition 2.1.1. *If $T : \mathcal{I}_F/\mathcal{P}_{\mathcal{F}} \to \prod_{\ell \neq p} \mathbb{Z}_\ell$ is a topological isomorphism, then*

$$T(\Phi_F \sigma \Phi_F^{-1}) = q^{-1} t(\sigma), \quad \sigma \in \mathcal{I}_F/\mathcal{P}_F$$

Proof. Any such topological isomorphism must differ from the one in 2.4 by multiplication by an element of $\prod_{\ell \neq p} \mathbb{Z}_\ell^\times$. So without loss of generality, we assume that T is the isomorphism in 2.4. Then it suffices check that the maps T_n (2.3) satisfy $T_n(\Phi\sigma\Phi^{-1}) = T_n(\sigma)^r$ for all $\sigma \in \mathrm{Gal}(K_n/F^{ur})$, where $rq \equiv 1 (\mathrm{mod}\ n)$, for an element $\Phi \in \mathrm{Gal}(K_n/F)$ which maps to $\Phi_F \in \mathrm{Gal}(F^{ur}/F)$.
Let $\alpha \in K_n$ such that $\alpha^n = \varpi$. Then $\Phi^{-1}(\alpha) = \zeta\alpha$, for some n^{th} root of unity $\zeta \in F^{ur}$. So we have,

$$\begin{aligned} T_n(\Phi\sigma\Phi^{-1}) &= \Phi(\sigma(\zeta))\Phi(\sigma(\alpha))/\alpha \\ &= \Phi(\zeta)\Phi(T_n(\sigma)\alpha)/\alpha \qquad (\zeta \in F^{ur}, T_n(\sigma)\alpha = \sigma(\alpha)) \\ &= \Phi(\zeta\alpha)\Phi(T_n(\sigma))/\alpha \\ &= \Phi(T_n(\sigma)) \end{aligned}$$

Note that reducing modulo the maximal ideal in the ring of integers for F^{ur} gives an isomorphism between the groups of n^{th} roots of unity of F^{ur} and its residue field $k_{F^{ur}}$. Since the Frobenius acts on the residue field by taking q^{th} powers, the inverse of the Frobenius takes n^{th} roots of unity in the residue field to their r^{th} powers. It follows that $\Phi(T_n(\sigma)) = T_n(\sigma)^r$. □

Hence, we have a fairly explicit description of $\Omega_F/\mathcal{P}_F$. Every finite extension of F^{tr} has p-power degree, since they "come from" totally wildly ramified extensions of tamely ramified extensions of F. Therefore, $\mathcal{P}_F = \mathrm{Gal}(\bar{F}/F^{tr})$ is a pro p-group; in fact, it is the unique pro p-Sylow subgroup of $\mathcal{I}_F$.

Next, we'll see a description of abelianisation of Ω_F.

2.2 Local Class Field Theory and the Weil Group

We state the local reciprocity law from local class field theory, which gives an essentially complete description of the abelian extensions of F. This will serve both as motivation for the definition of the Weil Group, and a tool to transfer Tate's theory of L-functions and local constants for GL_1 to characters of the Weil group. First, we define some notation:

Definition. For a topological group G, we denote by G^c its **commutator subgroup**, that is the (normal) subgroup generated by elements of the form $ghg^{-1}h^{-1}, g, h \in G$. The **abelianisation** of G, denoted by G^{ab}, is the largest (Hausdorff) abelian quotient of G, given by $G/\overline{G^c}$, that is, the quotient of G by the closure of its commutator.

Theorem 2.2.1 (Local Reciprocity)**.** *Let F be a non-Archimedean local field. There exists a continuous group homomorphism*

$$\theta_F : F^\times \to \mathrm{Gal}(F^{ab}/F) \cong \Omega_F^{ab}$$

where F^{ab} is the maximal abelian extension of F, determined by the following properties:

(i) θ_F composed with $\Omega_F^{ab} \xrightarrow{V_F} \widehat{\mathbb{Z}}$ is the negative of the valuation map $v_F : F^\times \to \mathbb{Z}$.

(ii) For any finite extension E of F contained in F^{ab}, if $\alpha \in F^\times$ is a norm from $K^\times$, then $\theta_F(\alpha) \in \mathrm{Gal}(F^{ab}/F)$ acts trivially on E.

It further satisfies the following properties:

(iii) θ_F maps U_F isomorphically onto $\mathrm{Gal}(F^{ab}/F^{ur}) = \mathcal{I}_F/\overline{\Omega_F^c}$, and U_F^1 onto $\mathrm{Gal}(F^{ab}/F^{ab} \cap F^{tr}) = \mathcal{P}_F\overline{\Omega_F^c}/\overline{\Omega_F^c}$.

(iv) Let $E \subset \bar{F}$ be a finite extension of F. Then the following diagrams commute:

$$\begin{array}{ccc} E^\times & \xrightarrow{\theta_E} & \Omega_E^{ab} \\ \downarrow{\scriptstyle N_{E/F}} & & \downarrow{\scriptstyle i} \\ F^\times & \xrightarrow{\theta_F} & \Omega_F^{ab} \end{array} \qquad (2.5) \qquad\qquad \begin{array}{ccc} F^\times & \xrightarrow{\theta_F} & \Omega_F^{ab} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ver}_{E/F}} \\ E^\times & \xrightarrow{\theta_E} & \Omega_E^{ab} \end{array} \qquad (2.6)$$

where i is the induced by the inclusion of Ω_E into Ω_F and $\mathrm{ver}_{E/F}$ is the transfer homomorphism or the "Verlagerung".

Remark. The negative sign in the property (i) of the reciprocity map shows up because in the definition of V_F we used the negative of usual isomorphism $\widehat{\mathbb{Z}} \cong \mathrm{Gal}(k_{\bar{F}}/k_F)$; we mapped 1 to the inverse of the Frobenius map $x \mapsto x^q$.

The proofs of the above statements can be found in the chapter 6 of [CF67]. The reciprocity map almost completely describes the maximal abelian quotient Ω_F^{ab} of Ω_F. To try to describe the non-abelian part, one considers representations of Ω_F.
The reciprocity map lets us view characters of Ω_F as characters of $F^\times$. The property (iii) of the map implies that the image of the reciprocity map is dense in $\mathrm{Gal}(F^{ab}/F)$, so no distinct characters end up getting identified. However, $F^\times$ has strictly more characters; the norm character cannot possibly come from a character of Ω_F, since the compactness of Ω_F forces all its characters to have bounded image. To account for such characters, we replace the Galois group with the Weil group.

Definition. The **Weil group** $\mathcal{W}_F$ of F (relative to $\bar{F}/F$) is the topological group with the underlying abstract group given by the subgroup $V_F^{-1}(\mathbb{Z})$ of Ω_F, and the initial topology such that the inclusion $i_F : \mathcal{W}_F \to \Omega_F$ and the induced map $V_F : \mathcal{W}_F \to \mathbb{Z}$ are continuous, where $\mathbb{Z}$ is seen as a discrete group. It is easy to see that this topology is determined by the following:

(i) $\mathcal{I}_F$ is an open subgroup of $\mathcal{W}_F$.

(ii) The topology on $\mathcal{I}_{\mathcal{F}}$ as a subspace of $\mathcal{W}_F$ agrees with its topology as the subgroup $\mathcal{I}_F = \mathrm{Gal}(\bar{F}/F^{ur}) \subset \Omega_F$.

Remark. As an abstract subgroup of Ω_F, $\mathcal{W}_F$ is generated by a Frobenius element Φ and $\mathcal{I}_F$. Its given topology, however, is not the subspace topology induced from Ω_F; in the subspace topology, any neighbourhood of identity contains must contain finite index subgroup, which the neighbourhood $\mathcal{I}_F$ does not. Moreover, $\mathcal{W}_F$ is locally profinite but not compact. The local profinite-ness follows from the profinite group $\mathcal{I}_F$ forming an open neighbourhood of identity. The lack of compactness is immediate from the fact that the map V_F restricts to give a surjective continuous map from $\mathcal{W}_F$ to the infinite discrete group $\mathbb{Z}$. Since the kernel of V_F is the compact open subgroup $\mathcal{I}_F$, we see that $\mathcal{W}_F$ is small in the sense of chapter 1.
The Weil groups behave much like Galois groups. The map $i_F : \mathcal{W}_F \to \Omega_F$ is a continuous homomorphism with a dense image (It is the inverse image of the dense subset $\mathbb{Z}$ under the open map V_F).

Proposition 2.2.2.

(i) Let E/F be a finite extension, $E \subset \bar{F}$. Denote by $\mathcal{W}_F^E$, the subgroup of $\mathcal{W}_F$ given by $i_F^{-1}(\Omega_E)$.

(a) *$\mathcal{W}_F^E$ is open and of finite index in $\mathcal{W}_F$, it is normal in $\mathcal{W}_F$ iff E/F is Galois.*

(b) *The natural inclusion $i : \Omega_E \to \Omega_F$ induces a topological isomorphism $\mathcal{W}_E \to \mathcal{W}_F$ with image $\mathcal{W}_F^E$.*

(c) *The canonical map $\mathcal{W}_F^E \backslash \mathcal{W}_F \to \Omega_E \backslash \Omega_F$ and the corresponding map for right cosets are a bijections. If E/F is Galois, these are group isomorphisms.*

(ii) *The map $E/F \mapsto \mathcal{W}_F^E$ is an inclusion reversing bijection between the set of finite subextensions of $\bar{F}/F$ and finite index open subgroups of $\mathcal{W}_F$.*

Proof. Recall that Ω_E embeds as an open subgroup of finite index in Ω_F. The first statement of (1a) follows immediately from injectivity and continuity of i_F. The normality statement of (1a) and (1c) follow from the density of the image of i_F.
For (1b), one has the commutative diagram:

$$\begin{array}{ccccc} \Omega_F & \xrightarrow{V_F} & \mathrm{Gal}(k_{\bar{F}}/k_F) & \xrightarrow{\sim} & \widehat{\mathbb{Z}} \\ {\scriptstyle i}\uparrow & & \uparrow & & \uparrow{\scriptstyle \times f} \\ \Omega_E & \xrightarrow{V_E} & \mathrm{Gal}(k_{\bar{F}}/k_E) & \xrightarrow{\sim} & \widehat{\mathbb{Z}} \end{array} \qquad (2.7)$$

where f is the degree of the extension k_E/k_F and i is the natural inclusion. Therefore, taking inverse images of 0 and $\mathbb{Z}$ from the top right corner of the diagram along all the maps, we see that i restricts to a continuous homomorphism $i_E(\mathcal{W}_E) \to i_F(\mathcal{W}_F)$ whose image is $\Omega_E \cap i_F(\mathcal{W}_F)$, such that $i^{-1}(\mathcal{I}_F) = \mathcal{I}_E$.

The map in $E/F \mapsto \mathcal{W}_F^E$ is clearly inclusion reversing. Its injectivity follows from observing that $\overline{i_F(\mathcal{W}_F^E)} = \Omega_E$. Now consider a finite index open subgroup H of $\mathcal{W}_F$. Motivated by the equality $\overline{i_F(\mathcal{W}_F^E)} = \Omega_E$, we consider the closure of $i_F(H)$ in Ω_F. Since the natural map $H\backslash\mathcal{W}_F \to \overline{i_F(H)}\backslash\Omega_F$ has dense image, $\overline{i_F(H)}$ has finite index in Ω_F. Therefore $\overline{i_F(H)} = \Omega_E$ for some finite extension E/F. We want to show that $H = \mathcal{W}_F^E$, or equivalently, $i_F(H) = V_F^{-1}(\mathbb{Z}) \cap \Omega_E$.
Consider the restriction of V_F to H and Ω_E. Let $\Phi_H \in H$ be such that $V_F(H) = V_F(\Phi_H)\mathbb{Z}$. Since H has finite index in $\mathcal{W}_F$, $V_F(H) \neq 0$, in particular, $V_F(\Phi_H) \neq 0$. Moreover, density of $i_F(H)$ in Ω_E implies that $V_F(\Omega_E) = V_F(\Phi_H)\widehat{\mathbb{Z}}$. Let $H_I = \mathcal{I}_F \cap H$, then $H = \Phi_H^{\mathbb{Z}} \ltimes H_I$. The map

$V_F(\Phi_H) \mapsto \Phi_H$ gives a section for $V_F : \Omega_E \to V_F(\Phi_H)\widehat{\mathbb{Z}}$. Let $\Phi_H^{\widehat{\mathbb{Z}}}$ denote the image of this section. This is the closure of $\Phi_H^{\mathbb{Z}}$ in Ω_E. We have $\Phi_H^{\widehat{\mathbb{Z}}} \cap H_I = 0$, since V_F is injective on $\Phi_H^{\widehat{\mathbb{Z}}}$. By continuity, $\Phi_H^{\widehat{\mathbb{Z}}}$ normalizes H_I, so the compact set $\Phi_H^{\widehat{\mathbb{Z}}} H_I$ is a subgroup, containing $i_F(H)$ as a dense subgroup. Then $\Omega_E = \overline{i_F(H)} = \Phi_H^{\widehat{\mathbb{Z}}} \ltimes H_I$. The equality $i_F(H) = \Phi_H^{\mathbb{Z}} H_I = V_F^{-1}(\mathbb{Z}) \cap \Omega_E$ immediately follows. □

The choice of topology for the Weil group is motivated by getting a cleaner restatement of the reciprocity law. First, a lemma;

Lemma 2.2.3. *The commutators $\mathcal{W}_F^c$ and Ω_F^c have the same closure. In particular induced map $i_F^{ab} : \mathcal{W}_F^{ab} \to \Omega_F^{ab}$ is injective.*

Proof. Observe that both $\mathcal{W}_F^c$ and Ω_F^c lie inside $\mathcal{I}_F$. Since $\mathcal{I}_F$ has the same topology as a subspace of both $\mathcal{W}_F$ and Ω_F, the density of the image of i_F implies that $\mathcal{W}_F^c$ is dense Ω_F^c. It follows that their closures coincide. □

From properties (i) and (iii) of the reciprocity map θ_F, it is easy to see that its image is exactly the subgroup $\mathcal{W}_F^{ab}$ of Ω_F. Property (iii) of the reciprocity law then implies that we get topological isomorphism from $F^\times$ to $\mathcal{W}_F^{ab}$. Consider the composite map,

$$\boldsymbol{a}_F : \mathcal{W}_F \longrightarrow \mathcal{W}_F^{ab} \xrightarrow{\sim} F^\times \xrightarrow{a \mapsto a^{-1}} F^\times$$

where the first map is abelianisation, second is the inverse of the isomorphism just obtained, and the last map is the multiplicative inversion.

Theorem 2.2.4 (Local Reciprocity)**.** *Let F be a non-Archimedean local field. There is a continuous group homomorphism,*

$$\boldsymbol{a}_F : \mathcal{W}_F \to F^\times$$

satisfying the following properties:

(i) The map $\boldsymbol{a}_F$ induces an isomorphism $\mathcal{W}_F^{ab} \cong F^\times$.

(ii) The composition of $\boldsymbol{a}_F$ with the valuation map $v_F : F^\times \to \mathbb{Z}$ is the restriction of V_F to Ω_F.

(iii) $\boldsymbol{a}_F(\mathcal{I}_F) = U_F$, $\boldsymbol{a}_F(\mathcal{P}_F) = U_F^1$.

(iv) Let $E \subset \bar{F}$ be a finite extension of F. Then the following diagrams commute:

$$\begin{array}{ccc} \mathcal{W}_E & \xrightarrow{\boldsymbol{a}_E} & E^\times \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle N_{E/F}} \\ \mathcal{W}_F & \xrightarrow{\boldsymbol{a}_F} & F^\times \end{array} \tag{2.8}$$

$$\begin{array}{ccc} \mathcal{W}_F^{ab} & \xrightarrow{\theta_F} & F^\times \\ \big\downarrow{\scriptstyle \mathrm{ver}_{E/F}} & & \big\downarrow \\ \mathcal{W}_E^{ab} & \xrightarrow{\theta_E} & E^\times \end{array} \tag{2.9}$$

where i is the induced by the inclusion of Ω_E into Ω_F as in (1b) of Proposition 2.2.2, and $\mathrm{ver}_{E/F}$ is the transfer homomorphism.

Moreover, these properties characterize $\boldsymbol{a}_F$.

All the statements follow without much work from the Galois group version of the reciprocity law and the discussion above. We point out that the composition with multiplicative inversion was only to avoid a sign discrepancy in property (ii).
It is immediate from this formulation of the reciprocity law that the map $\chi \mapsto \chi \circ \boldsymbol{a}_F$ gives an isomorphism from group of characters of $F^\times$ to $\mathcal{W}_F$.

We now discuss some generalities about smooth representations of $\mathcal{W}_F$.

2.3 Smooth representations of the Weil Group

As mentioned in the last section, unlike Ω_F, the Weil group $\mathcal{W}_F$ is not compact, so it can have non-semisimple smooth representations. However, irreducible representations of $\mathcal{W}_F$ are almost the same as the ones for Ω_F.

Lemma 2.3.1. *Let (ρ, V) be an irreducible smooth representation of $\mathcal{W}_F$.*

(i) For any Frobenius element $\Phi \in \mathcal{W}_F$, there exists a positive integer d such that $\rho(\Phi)^d$ is multiplication by a scalar.

(ii) V is finite dimensional.

Proof. Let $v \in V, v \neq 0$. Then v is fixed by an open subgroup $\mathcal{J}$ of $\mathcal{I}_F$. There exists a finite Galois extension E of F such that $\Omega_E \cap \mathcal{I}_F \subset \mathcal{J}$. Replacing $\mathcal{J}$ by $\Omega_E \cap \mathcal{I}_F$ we can assume that $\mathcal{J}$ is normal in $\mathcal{W}_F$. Since V is irreducible, it is spanned by $\mathcal{W}_F$ translates of v, hence $\mathcal{J} \subset \ker \rho$. In particular, $\ker \rho$ is open.
Let Φ be a Frobenius element. Since $\ker \rho$ is open, $\rho(\mathcal{I}_F)$ is finite. The conjugation action of $\rho(\Phi)$ on $\rho(\mathcal{I}_F)$ must then have finite order; in other words $\rho(\Phi)^d$ commutes with $\rho(\mathcal{I}_F)$ for some positive integer d. Since $\mathcal{W}_F$

is generated by $\mathcal{I}_F$ and Φ, this further implies that $\rho(\Phi)^d$ commutes with $\rho(\mathcal{W}_F)$, that is, $\rho(\Phi)^d \in \mathrm{End}_{\mathcal{W}_F}(V)$. By Schur's lemma, we get that $\rho(\Phi)^d$ is a scalar. It follows that V is spanned by $\{\rho(\Phi^i x)v \mid 0 \leq i < d, x \in \mathcal{I}_F/(\mathcal{I}_F \cap \ker \rho)\}$ and is in particular finite dimensional. □

We record a weaker version of (i) applicable to all finite dimensional smooth representations of $\mathcal{W}_F$.

Lemma 2.3.2. *Let (ρ, V) be a finite dimensional smooth representation of $\mathcal{W}_F$. For any Frobenius element Φ, there exists a positive integer d such that $\rho(\Phi)^d$ is commutes with $\rho(\mathcal{W}_F)$.*

Proof. Finite-dimensionality and compactness of $\mathcal{I}_F$ imply that $\rho(\mathcal{I}_F)$ is finite. Rest of the argument proceeds as in the proof of the previous lemma. □

Given a smooth representation ρ of Ω_F, we get a smooth representation $\rho \circ i_F$ of $\mathcal{W}_F$, which can be seen as the "restriction" of ρ to $\mathcal{W}_F$.

Proposition 2.3.3. *(i) Let ρ be a finite dimensional smooth representation of Ω_F. Then $\rho(\Omega_F)$ is finite and is equal to $\rho \circ i_F(\mathcal{W}_F)$.*

(ii) The map $\rho \mapsto \rho \circ i_F$ gives a fully faithful functor from the category $\mathrm{Rep}^f(\Omega_F)$ to $\mathrm{Rep}^f(\mathcal{W}_F)$. That is,

$$\mathrm{Hom}_{\Omega_F}(\rho_1, \rho_2) = \mathrm{Hom}_{\mathcal{W}_F}(\rho_1 \circ i_F, \rho_2 \circ i_F)$$

for $(\rho_i, V_i) \in \mathrm{Rep}^f(\Omega_F)$. In particular ρ_i are isomorphic iff $\rho_i \circ i_F$ are isomorphic.

(iii) Let (ρ, V) be a finite dimensional smooth representation of Ω_F. Then ρ is irreducible iff $\rho \circ i_F$ is irreducible.

Proof. (i) Profiniteness of Ω_F implies that $\ker \rho$ is open and $\rho(\Omega_F)$ is finite. It then follows from the density of $i_F(\mathcal{W}_F)$ in Ω_F that the open (and hence closed) subgroup $i_F(\mathcal{W}_F) \cdot \ker \rho$ is all of Ω_F. Therefore $\rho(\Omega_F) = \rho \circ i_F(\mathcal{W}_F)$.

(ii) Since any Ω_F-map is also a $\mathcal{W}_F$-map, the map $\rho \mapsto \rho \circ i_F$ defines a functor from $\mathrm{Rep}^f(\Omega_F)$ to $\mathrm{Rep}^f(\mathcal{W}_F)$. To show full-faithfulness, it suffices to show that any $\mathcal{W}_F$-map $f : V_1 \to V_2$ is also an Ω_F-map. Consider such $\mathcal{W}_F$-map $f : V_1 \to V_2$. As in the proof of (i), we have $\Omega_F = i_F(\mathcal{W}_F) \cdot (\ker \rho_1 \cap \ker \rho_2)$. Let $x = i_F(w)k \in \Omega_F$, where $w \in \mathcal{W}_F, k \in \ker \rho_1 \cap \ker \rho_2$. Then,

$$f(\rho_1(x)v) = f(\rho_1 \circ i_F(w)v) = f(\rho) = \rho_2 \circ i_F(w)f(v) = \rho_2(x)f(v)$$

for any $v \in V_1$. Therefore f is also Ω_F-linear.

(iii) From (i) it follows that a subspace W of V is a Ω_F-subspace iff it is a $\mathcal{W}_F$-subspace. The statement of (iii) immediately follows.

□

Definition. We say a finite dimensional smooth representation τ of $\mathcal{W}_F$ is of **Galois type**, if $\tau = \rho \circ i_F$ for some (finite-dimensional) smooth representation ρ of Ω_F.

It follows from part (i) of the proposition that any finite dimensional representation of $\mathcal{W}_F$ of Galois type has finite image. The converse is also true:

Theorem 2.3.4. *Let (ρ, V) be an finite dimensional smooth representation of $\mathcal{W}_F$. The following are equivalent:*

(i) ρ is of Galois type.

(ii) ρ has finite image.

(iii) $\rho(\Phi)^d = \mathrm{Id}$ for some positive integer d and a Frobenius element Φ.

Proof. The implication (i) $\Longrightarrow$ (ii) follows from the previous proposition, (ii) $\Longrightarrow$ (iii) is immediate. We show (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i).

Let $\rho(\Phi)^d = \mathrm{Id}$ for some positive integer d and a Frobenius element Φ. Since V is finite dimensional, $\rho(\mathcal{I}_F)$ must be finite. Moreover, $\rho(\mathcal{I}_F)$ is normal in $\rho(W_F)$, which means $\rho(\mathcal{W}_F) = \{\rho(\Phi^i x) \mid 0 \leq i \leq d, x \in \mathcal{I}_F\}$ is finite. Thus (iii) $\Longrightarrow$ (ii).

Lastly, if $\rho(\mathcal{W}_F)$ is finite, $\ker \rho$ is has finite index in $\mathcal{W}_F$. Since V is finite dimensional, it is also an open subgroup. Therefore, $\ker \rho = \mathcal{W}_F^E$ for some finite extension E/F. But i_F induces $\mathcal{W}_F/\mathcal{W}_F^E \cong \Omega_F/\Omega_E$. By inverting and composing we obtain an irreducible smooth representation τ of Ω_F on V, which satisfies $\rho = \tau \circ i_F$. So (ii) $\Longrightarrow$ (i). □

In fact, irreducible smooth representations of $\mathcal{W}_F$ are essentially the same as those of Ω_F. First, a definition:

Definition. We call a representation of $\mathcal{W}_F$ or Ω_F **unramified**, if it is trivial on $\mathcal{I}_F$.

Note that for characters, this is consistent with the notion of unramified characters on $F^\times$ (see here) via the property (iii) of the reciprocity map. An example of such a character is the norm character,

$$\|x\| := \|\boldsymbol{a}_F(x)\| = q^{-V_F(x)}, \quad x \in \mathcal{W}_F$$

where the $\|\boldsymbol{a}_F(x)\|$ is the absolute value of $\boldsymbol{a}_F(x) \in F^\times$. The norm character is clearly not of Galois type. Since any unramified character factors through $\mathcal{W}_F/\mathcal{I}_F \cong \mathbb{Z}$, one gets the following:

Lemma 2.3.5. *Any unramified character of $\mathcal{W}_F$ is of the form $\|\cdot\|^s$, for $s \in \mathbb{C}$.*

Turns out that an arbitrary irreducible representation of $\mathcal{W}_F$ is of Galois type, upto twisting by an unramified character:

Corollary 2.3.5.1. *Let (ρ, V) be an irreducible smooth representation of $\mathcal{W}_F$. Then $\rho = \chi \otimes \tau$ for some unramified character χ and an irreducible smooth representation τ of Galois type.*

Proof. By lemma 2.3.1, ρ is finite dimensional and $\rho(\Phi)^d = \lambda\,\mathrm{Id}$ for some positive integer d. Consider an unramified character such that $\chi(\Phi)^d = \lambda$ (for example, $\chi = \|\cdot\|^c$ where c satisfies $q^{-cd} = \lambda$). Then $\tau := \chi^{-1} \otimes \rho$ satisfies condition (iii) of the previous proposition. □

Having understood irreducible representations of the Weil group, we tackle the issue of semisimplicity:

Proposition 2.3.6. *Let (ρ, V) be a finite dimensional smooth representation of $\mathcal{W}_F$, and $\Phi \in \mathcal{W}_F$ be a Frobenius element. The following are equivalent:*

(i) *ρ is a semisimple representation*

(ii) *$\rho(\Phi) \in \mathrm{Aut}_{\mathbb{C}}(V)$ is semisimple*

(iii) *$\rho(\Psi) \in \mathrm{Aut}_{\mathbb{C}}(V)$ is semisimple for all $\Psi \in \mathcal{W}_F$.*

Proof. (iii) $\Longrightarrow$ (ii) is clear, we show (ii) $\Longrightarrow$ (i) and (i) $\Longrightarrow$ (iii)
Let $\rho(\Phi)$ be semisimple. By lemma 2.3.2, $\rho(\Phi)^d$ commutes with $\rho(\mathcal{W}_F)$ for some positive integer d. Consider the finite index subgroup $H = \langle \Phi^d, \mathcal{I}_F \rangle \subset \mathcal{W}_F$. Since $\rho(\Phi^d)$ is semisimple, the space V decomposes into a direct sum of eigenspaces $V = \oplus_\lambda V_\lambda$ for $\rho(\Phi^d)$, and since $\rho(\Phi^d)$ commutes with $\rho(H)$, these eigenspaces are H-subrepresentations. Since $\rho(\Phi^d)$ acts as a scalar on V_λ, a subspace of V_λ is an H-subspace iff it is a $\mathcal{I}_F$-subspace. It follows

that V_λ is H-semisimple iff it is $\mathcal{I}_F$-semisimple, which it must be since $\mathcal{I}_F$ is profinite. Therefore each V_λ, and hence V is H-semisimple. It follows that V is $\mathcal{W}_F$-semisimple since H has finite index in $\mathcal{W}_F$.
Now we assume that ρ is semisimple and pick $\Psi \in \mathcal{W}_F$. Then it decomposes as a direct sum of irreducible representations of $\mathcal{W}_F$. So we reduce to the case where ρ is irreducible. But by the previous corollary, we know $\rho = \chi \otimes \rho'$, for an irreducible smooth representation ρ' of Galois type, and an unramified character χ. Since $\rho(\Psi) = \chi(\Psi)\rho'(\Psi)$ is semisimple iff $\rho'(\Psi)$ is semisimple, we reduce to considering irreducible smooth representations ρ of Galois type. But such representations have finite image, so $\rho(\Psi)^d = \mathrm{Id}$ for some positive integer d. Therefore $\rho(\Psi)$ is semisimple. □

Let E/F be a finite separable extension contained in our fixed algebraic closure $\bar{F}$. Then we have functors $\mathrm{Res}_{E/F} : \rho \mapsto \rho\,|_{\mathcal{W}_E}$ and $\mathrm{Ind}_{E/F} : \rho \mapsto \mathrm{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F} \rho$, where we identify $\mathcal{W}_E$ with the finite index open subgroup $\mathcal{W}_F^E$ of $\mathcal{W}_F$ (see Proposition 2.2.2). Then the lemmas 1.2.9, 1.2.16 and 1.2.17 immediately give the following:

Lemma 2.3.7. *Let E/F be a finite separable extension contained in $\bar{F}$.*

(i) A smooth representation ρ of $\mathcal{W}_F$ is semisimple iff the representation $\mathrm{Res}_{E/F}\,\rho$ of $\mathcal{W}_E$ is semisimple.

(ii) A smooth representation τ of $\mathcal{W}_E$ is semisimple iff the representation $\mathrm{Ind}_{E/F}\,\tau$ of $\mathcal{W}_F$ is semisimple.

2.4 A larger class of representations

In practice, most representations of Ω_F and $\mathcal{W}_F$ are ℓ-adic representations. Particularly important instances of these are étale cohomology groups, which includes of course, one of our objects of interest, the Tate module of an elliptic curve. We start by defining ℓ-adic representations.

Let K be a topological field. Then for any finite dimensional vector space V over K, a choice of basis gives a isomorphisms $V \cong K^d$ and $\mathrm{Aut}_K(V) \cong \mathrm{GL}_d(K) \subset \mathrm{M}_d(K) \cong K^{d^2}$, where $d = \dim_K V$. This allows us to endow both V and $\mathrm{Aut}_K(V)$ with topologies. It is not too difficult to show that these are independent of the choice of basis for V.

Definition. Let G be a topological group. By a **continuous representation** of G over K we mean a finite dimensional representation (π, V) of G

over K such that the action map $a_\pi : G \times V \to V$ given by $(g, v) \mapsto \pi(g)v$ for $g \in G, v \in V$ is continuous. These form a category $\text{c-Rep}_K^f(G)$ with G-linear maps as morphisms.
It can be shown that the continuity condition is equivalent to $\pi : G \to \text{Aut}_K(V)$ being continuous.

Using lemma 1.2.2, it is easy to see that the any finite dimensional smooth representation over K is also a continuous representation over K, in fact smooth representations are exactly continuous representations when the topology on K is discrete.

For the rest of this section, we take K to be an algebraic extension of $\mathbb{Q}_\ell$ with the ℓ-adic topology given by the unique extension of the ℓ-adic valuation on $\mathbb{Q}_\ell$ to a $\mathbb{Q}$-valued valuation on K.

Definition. Finite dimensional, continuous representations over algebraic extensions of $\mathbb{Q}_\ell$ with the ℓ-adic topology are called **ℓ-adic representations**.

As we will see later, $\mathcal{I}_F$ can have infinite image under ℓ-adic representations of $\mathcal{W}_F$, something which cannot happen within the realm of smooth representations. However, we will see that we can still classify ℓ-adic representations of the Weil group using their smooth representations, if we allow ourselves to carry a bit more data with them. We only discuss the case $\ell \neq p$, the residue characteristic of F.

We start with introducing some tools which will be useful to us. Let V be a finite dimensional vector space over some field. For a nilpotent element $\mathfrak{n} \in \text{End}(V)$ and the unipotent element $\mathfrak{u} = 1 + \mathfrak{n}$, set,

$$\exp \mathfrak{n} = 1 + \sum_{i \geq 1} \frac{\mathfrak{n}^i}{i!}, \qquad \log \mathfrak{u} = \sum_{j \geq 1} (-1)^{j-1} \frac{\mathfrak{n}^j}{j}.$$

Both sums have finitely many non-zero terms, and the expected identities $\log(\exp \mathfrak{n}) = \mathfrak{u}$ and $\exp(\log \mathfrak{u}) = \mathfrak{n}$ hold. Moreover if $\mathfrak{g} \in \text{Aut}(V)$, then $\mathfrak{g}(\exp \mathfrak{n})\mathfrak{g}^{-1} = \exp(\mathfrak{g}\mathfrak{n}\mathfrak{g}^{-1})$, and if $\mathfrak{n}, \mathfrak{n}' \in \text{End}(V)$ are commuting nilpotent elements, $\mathfrak{n} + \mathfrak{n}'$ is also nilpotent, and we have $\exp(\mathfrak{n} + \mathfrak{n}') = (\exp \mathfrak{n})(\exp \mathfrak{n}')$.

We also need the following lemma:

Lemma 2.4.1. *There exists a continuous surjective homomorphism $t : \mathcal{I}_F \to \mathbb{Z}_\ell$, unique upto multiplication by an element of $\mathbb{Z}_\ell^\times$. Any such t satisfies*

$$t(gxg^{-1}) = \|g\| t(x), \quad x \in \mathcal{I}_F, g \in \mathcal{W}_F. \tag{2.10}$$

Proof. Recall that there is a topological isomorphism $T : \mathcal{I}_F/\mathcal{P}_F \to \prod_{q\neq p}\mathbb{Z}_q$. This gives a continuous surjection, $t = \pi_\ell \circ T \circ \tau : \mathcal{I}_F \to \mathbb{Z}_\ell$, where π_ℓ is the projection $\prod_{q\neq p}\mathbb{Z}_q \to \mathbb{Z}_\ell$ and $\tau : \mathcal{I}_F \to \mathcal{I}_F/\mathcal{P}_F$ is the quotient map. The equation 2.10 follows from proposition 2.1.1.

Moreover, since $\mathcal{P}_F$ is a pro p-group and $\ell \neq p$, any surjection $t' : \mathcal{I}_F \to \mathbb{Z}_\ell$ factors $\mathcal{I}_F/\mathcal{P}_F$. It further factors through $\pi_\ell \circ T$ (all finite quotients of $\prod_{q\neq p,\ell}\mathbb{Z}_q$ have order prime to ℓ, that is, it has pro order prime to ℓ). Therefore t' differs from t by an automorphism of $\mathbb{Z}_\ell$, that is, by multiplication by an element of $\mathbb{Z}_\ell^\times$. It follows that t' satisfies 2.10 since t does. □

Now we can state the result which makes studying ℓ-adic representations via smooth representations possible:

Theorem 2.4.2. *Let (σ, V) be an ℓ-adic representation of $\mathcal{W}_F$ over K, with $\ell \neq p$. Furthermore, let $t : \mathcal{I}_F \to \mathbb{Z}_\ell$ be a continuous surjective homomorphism. There is a unique nilpotent endomorphism $\mathfrak{n}_{\sigma,t} \in \mathrm{End}_K(V)$ such that*

$$\sigma(x) = \exp(t(x)\mathfrak{n}_{\sigma,t}), \tag{2.11}$$

for all x in some open subgroup of $\mathcal{I}_F$.

See [BH06, Theorem 32.5] for a proof.

Corollary 2.4.2.1. *Let (σ, V), $t : \mathcal{I}_F \to \mathbb{Z}_\ell$ and $\mathfrak{n}_{\sigma,t}$ be as in the theorem.*

(i) The endomorphism $\mathfrak{n}_{\sigma,t}$ satisfies:

$$\sigma(g)\mathfrak{n}_{\sigma,t}\sigma(g)^{-1} = \|g\|\mathfrak{n}_{\sigma,t}, \quad g \in \mathcal{W}_F$$

In particular, for $x \in \mathcal{I}_F$, $\sigma(x)$ commutes with $\mathfrak{n}_{\sigma,t}$.

(ii) If $t' = \alpha t : \mathcal{I}_F \to \mathbb{Z}_\ell$ another continuous surjective homomorphism, where $\alpha \in \mathbb{Z}_\ell^\times$, then $\mathfrak{n}_{\sigma,t'} = \frac{1}{\alpha}\mathfrak{n}_{\sigma_\Phi,t}$.

Proof. We know that (2.11) holds for x in some open subgroup H of $\mathcal{I}_F$. Since $\mathcal{I}_F$ is a subspace of Ω_F, we may assume H is normal in Ω_F and hence in $\mathcal{W}_F$. Then we have,

$$\begin{aligned}\sigma(x) &= \sigma(g)\sigma(gxg^{-1})\sigma(g)^{-1}\\ &= \sigma(g)\exp(t(gxg^{-1})\mathfrak{n}_{\sigma,t})\sigma(g)^{-1}\\ &= \sigma(g)\exp(\|g\|^{-1}t(x)\mathfrak{n}_{\sigma,t})\sigma(g)^{-1}\\ &= \exp(t(x)\|g\|^{-1}\sigma(g)\mathfrak{n}_{\sigma,t}\sigma(g)^{-1})\end{aligned}$$

for any $x \in H, g \in \mathcal{W}_F$, using (2.10). By uniqueness of $\mathfrak{n}_{\sigma,t}$, (i) follows. For (ii), note that (2.11) gives,

$$\sigma(x) = \exp(t(x)\mathfrak{n}_{\sigma_{\Phi,t}}) = \exp\left(t'(x)\frac{1}{\alpha}\mathfrak{n}_{\sigma_{\Phi,t}}\right)$$

for x in an open subgroup of $\mathcal{I}_F$. Therefore $\frac{1}{\alpha}\mathfrak{n}_{\sigma_{\Phi,t}}$ satisfies the defining property of $\mathfrak{n}_{\Phi,t'}$. □

We can now construct a smooth representation $(\sigma, V, \mathfrak{n}_{\sigma_{\Phi,t}})$ using the following lemma,

Lemma 2.4.3. *Let (σ, V) be a finite dimensional (abstract) representation of $\mathcal{W}_F$ over K, and $\mathfrak{n} \in \mathrm{End}_K(V)$ be a nilpotent element such that*

$$\sigma(g)\mathfrak{n}\sigma(g)^{-1} = \|g\|\mathfrak{n}, \quad g \in \mathcal{W}_F \tag{2.12}$$

Then, for a Frobenius element $\Phi \in \mathcal{W}_F$ and a continuous surjective homomorphism $t : \mathcal{I}_F \to \mathbb{Z}_\ell$,the map

$$\sigma_{\Phi,t,\mathfrak{n}}(\Phi^a x) = \sigma(\Phi^a x)\exp(-t(x)\mathfrak{n}), \quad a \in \mathbb{Z}, x \in \mathcal{I}_F$$

is a homomorphism, that is $\sigma_{\Phi,t,\mathfrak{n}}$ is a representation of $\mathcal{W}_F$ on V. Moreover, $\sigma_{\Phi,t,\mathfrak{n}}$ satisfies,

$$\sigma_{\Phi,t,\mathfrak{n}}(g)\mathfrak{n}\sigma_{\Phi,t,\mathfrak{n}}(g)^{-1} = \|g\|\mathfrak{n}, \quad g \in \mathcal{W}_F. \tag{2.13}$$

Proof. First, we note that the map $\sigma_{\Phi,t,\mathfrak{n}}$ is well defined since every element of $\mathcal{W}_F$ can be uniquely written as $\Phi^a x$ for $a \in \mathbb{Z}, x \in \mathcal{I}_F$. Now for $g = \Phi^a x, h = \Phi^b y \in \mathcal{W}_F$,

$$\begin{aligned}
\sigma_{\Phi,t,\mathfrak{n}}(gh) &= \sigma_{\Phi,t,\mathfrak{n}}(\Phi^a x \Phi^b y) = \sigma_{\Phi,t,\mathfrak{n}}(\Phi^{a+b}x'y) \quad \text{where } x' = \Phi^{-b}x\Phi^b \in \mathcal{I}_F \\
&= \sigma(\Phi^{a+b}x'y)\exp(-t(x'y)\mathfrak{n}) \\
&= \sigma(\Phi^a x)\sigma(\Phi^b y)\exp(-t(x')\mathfrak{n})\exp(-t(y)\mathfrak{n}) \\
&= \sigma(\Phi^a x)\exp(-t(\Phi^{-b}x\Phi^b)\sigma(h)\mathfrak{n}\sigma(h)^{-1})\sigma(\Phi^b y)\exp(-t(y)\mathfrak{n}) \\
&= \sigma_{\Phi,t,\mathfrak{n}}(g)\sigma_{\Phi,t,\mathfrak{n}}(h)
\end{aligned}$$

using (2.10), (2.12) and $\|h\| = \|\Phi^b\|$. Finally, (2.13) follows easily from (2.12) and the observation that $\mathfrak{n}$ commutes with $\exp(t\mathfrak{n})$ for any $t \in \mathbb{Q}_\ell$ and a nilpotent element $\mathfrak{n}$. □

Corollary 2.4.3.1. *Let (σ, V), $t : \mathcal{I}_F \to \mathbb{Z}_\ell$, and $\mathfrak{n}_{\sigma,t}$ be as in theorem 2.4.2. Then the representation $\sigma_\Phi := \sigma_{\Phi,t,\mathfrak{n}_{\sigma,t}}$ as in the previous lemma is smooth and independent of choice of t. Moreover it satisfies,*

$$\sigma_\Phi(g)\mathfrak{n}_{\sigma,t}\sigma_\Phi(g)^{-1} = \|g\|\mathfrak{n}_{\sigma,t}, \quad g \in \mathcal{W}_F \tag{2.14}$$

Proof. By corollary 2.4.2.1(i), the previous lemma is applicable and hence $\sigma_{\Phi,t,\mathfrak{n}_{\sigma,t}}$ gives a representation of $\mathcal{W}_F$ on V over K. But by definition of $\mathfrak{n}_{\sigma,t}$, this representation is trivial on an open subset of $\mathcal{I}_F$ and hence $\mathcal{W}_F$, so it must be smooth. Moreover, the independence from the choice of t is clear from corollary 2.4.2.1(ii), and (2.14) is exactly (2.13) from the previous lemma. □

Remark. One can recover the ℓ-adic representation (σ, V) from the data $(\sigma_\Phi, V, \mathfrak{n}_{\sigma,t})$; in the notation of lemma 2.4.3,

$$\sigma(\Phi^a x) = \tau_{\Phi,t,-\mathfrak{n}}(\Phi^a x) = \sigma_\Phi(\Phi^a x)\exp(t(x)\mathfrak{n}_{\sigma,t})$$

where $\tau = \sigma_\Phi$ and $\mathfrak{n} = \mathfrak{n}_{\sigma,t}$. So we enlarge the category of smooth representations of $\mathcal{W}_F$ to account for the extra information carried by the nilpotent element $\mathfrak{n}_{\sigma,t}$.

Definition. A **Weil-Deligne representation** over K, is a triple $(\rho, V, \mathfrak{n})$, where (ρ, V) is a finite dimensional smooth representation of $\mathcal{W}_F$ over K, and $\mathfrak{n} \in \mathrm{End}_K(V)$ is a nilpotent endomorphism satisfying:

$$\rho(x)\mathfrak{n}\rho(x)^{-1} = \|g\|\mathfrak{n}, \quad x \in \mathcal{W}_F \tag{2.15}$$

These representations form a category D-$\mathrm{Rep}_K(\mathcal{W}_F)$, with morphisms $f : (\rho_1, V_1, \mathfrak{n}_1) \to (\rho_2, V_2, \mathfrak{n}_2)$ given by a $\mathcal{W}_F$-linear map $f : V_1 \to V_2$, which satisfies $f \circ \mathfrak{n}_1 = \mathfrak{n}_2 \circ f$.

It is this larger category of representations of $\mathcal{W}_F$, which forms the other side of the Local Langlands Correspondence for GL_n.

The corollary 2.4.3.1 then says exactly that the triple $(\sigma_\Phi, V, \mathfrak{n}_{\sigma,t})$ constructed using an ℓ-adic representation (σ, V) is a Weil-Deligne representation over K. In fact, we can say more:

Theorem 2.4.4. *Let $\Phi \in \mathcal{W}_F$ be a Frobenius element and $t : \mathcal{I}_F \to \mathbb{Z}_\ell$ be a continuous surjective homomorphism. The map $(\sigma, V) \mapsto (\sigma_\Phi, V, \mathfrak{n}_{\sigma,t})$ is functorial, and induces an equivalence of categories*

$$D_{\Phi,t} : \text{c-}\mathrm{Rep}_K^f(\mathcal{W}_F) \to \text{D-}\mathrm{Rep}_K(\mathcal{W}_F)$$

Moreover,

(i) If Φ' is another Frobenius element, then the functors $D_{\Phi,t}$ and $D_{\Phi',t}$ are naturally isomorphic, that is, there is a family of isomorphisms,

$$A_{\Phi,\Phi',\sigma,t} : (\sigma_\Phi, V, \mathfrak{n}_{\sigma,t}) \xrightarrow{\sim} (\sigma_{\Phi'}, V, \mathfrak{n}_{\sigma,t})$$

for $(\sigma, V) \in$ c-$\mathrm{Rep}_K^f(\mathcal{W}_F)$, which is natural in (σ, V).

(ii) Assume K is algebraically closed. If $t' : \mathcal{I}_F \to \mathbb{Z}_\ell$ is another continuous surjective homomorphism, then $D_{\Phi,t}(\sigma, V)$ and $D_{\Phi,t'}(\sigma, V)$ are isomorphic.

Therefore, if K is algebraically closed, the functor $D_{\Phi,t}$ induces a bijection on isomorphism classes of n-dimensional ℓ-adic representations of $\mathcal{W}_F$ over K and n-dimensional Weil-Deligne representations over K, independent of the choices of Φ and t.

Proof. **Establishing Functoriality of $D_{\Phi,t}$:**
We've already seen that $(\sigma_\Phi, V, \mathfrak{n}_{\sigma,t})$ is a Weil-Deligne representation. To show functoriality, we prove that if $f : (\sigma, V) \to (\rho, W)$ is a $\mathcal{W}_F$-linear map between ℓ-adic representations over K, then f is also morphism of Weil-Deligne representations from $(\sigma_\Phi, V, \mathfrak{n}_{\sigma,t})$ to $(\rho_\Phi, W, \mathfrak{n}_{\rho,t})$. For x in an open subgroup H of $\mathcal{I}_F$, we have,

$$f \circ \exp(t(x)\mathfrak{n}_{\sigma,t}) = f \circ \sigma(x) = \rho(x) \circ f = \exp(t(x)\mathfrak{n}_{\rho,t}) \circ f$$

By expanding the series for log, we get,

$$\begin{aligned} f \circ (t(x)\mathfrak{n}_{\sigma,t}) &= f \circ \log(\exp(t(x)\mathfrak{n}_{\sigma,t})) \\ &= \log(\exp(t(x)\mathfrak{n}_{\rho,t})) \circ f \\ &= (t(x)\mathfrak{n}_{\rho,t}) \circ f \end{aligned}$$

The surjection t cannot be trivial on the open subgroup H since its image $\mathbb{Z}_\ell$ is not discrete. Therefore there exists $x \in H$ such that $t(x) \neq 0$, so by the computation above we get,

$$f \circ \mathfrak{n}_{\sigma,t} = \mathfrak{n}_{\rho,t} \circ f \tag{2.16}$$

The fact that $f \circ \sigma_\Phi(g) = \rho_\Phi(g) \circ f$ for all $g \in \mathcal{W}_F$ then follows easily from this and that $f \circ \sigma(g) = \rho(g) \circ f$ for all $g \in \mathcal{W}_F$.

Constructing an inverse to $D_{\Phi,t}$:
To show that $D_{\Phi,t}$ is an equivalence of categories, we show that

$C_{\Phi,t} : (\rho, V, \mathfrak{n}) \mapsto (\rho_{\Phi,t,-\mathfrak{n}}, V)$ (see lemma 2.4.3) gives the inverse functor; first we need to show $(\rho_{\Phi,t,-\mathfrak{n}}, V) \in \text{c-Rep}_K^f(\mathcal{W}_F)$. We know that $\rho_{\Phi,t,-\mathfrak{n}}$ is a homomorphism from lemma 2.4.3, and its continuity is an easy consequence of continuity of ρ and of the map $x \mapsto \exp(t(x)\mathfrak{n})$.
Moreover, if $g : (\rho_1, V_1, \mathfrak{n}_1) \to (\rho_2, V_2, \mathfrak{n}_2)$ is a morphism of Weil-Deligne representations, then it follows easily from definitions that g is also a morphism from $((\rho_1)_{\Phi,t,-\mathfrak{n}_1}, V_1)$ to $((\rho_2)_{\Phi,t,-\mathfrak{n}_2}, V_2)$.
Lastly, if $(\sigma, V) \in \text{c-Rep}_K^f(\mathcal{W}_F)$, then the remark above says $C_{\Phi,t} \circ D_{\Phi,t}(\sigma, V) = (\sigma, V)$. Conversely if $(\rho, V, \mathfrak{n}) \in \text{D-Rep}_K(\mathcal{W}_F)$, then for x in the open subgroup $\ker \rho \cap \mathcal{I}_F$,

$$\rho_{\Phi,t,-\mathfrak{n}}(x) = \exp(t(x)\mathfrak{n})$$

Therefore by the uniqueness part of theorem 2.4.2, $\mathfrak{n}_{\sigma,t} = \mathfrak{n}$, where $\sigma = \rho_{\Phi,t,-\mathfrak{n}}$. It then easily follows that $D_{\Phi,t} \circ C_{\Phi,t}(\rho, V, \mathfrak{n}) = (\rho, V, \mathfrak{n})$. Lastly, both the functors don't do anything to morphisms. This finishes the proof that $D_{\Phi,t}$ is an equivalence of categories.

Proof of claim (i)
If Φ' is another Frobenius element, then $\Phi' = \Phi y$ for some $y \in \mathcal{I}_F$. Set

$$A_{\Phi,\Phi',\sigma,t} = \exp\left(\frac{1}{q-1}t(y)\mathfrak{n}_{\sigma,t}\right) \in \text{Aut}_K(V)$$

For $x \in \mathcal{I}_F$, using corollary 2.4.2.1(i),

$$\begin{aligned}
A_{\Phi,\Phi',\sigma,t} \circ \sigma_\Phi(x) &= \exp\left(\frac{1}{q-1}t(y)\mathfrak{n}_{\sigma,t}\right)\sigma(x)\exp(-t(x)\mathfrak{n}_{\sigma,t}) \\
&= \sigma(x)\exp\left(\frac{1}{q-1}t(y)\mathfrak{n}_{\sigma,t}\right)\exp(-t(x)\mathfrak{n}_{\sigma,t}) \\
&= \sigma(x)\exp(-t(x)\mathfrak{n}_{\sigma,t})\exp\left(\frac{1}{q-1}t(y)\mathfrak{n}_{\sigma,t}\right) \\
&= \sigma_{\Phi'}(x) \circ A_{\Phi,\Phi',\sigma,t}
\end{aligned}$$

Moreover, corollary 2.4.2.1(i) also gives,

$$\begin{aligned}
A_{\Phi,\Phi',\sigma,t} \circ \sigma_\Phi(\Phi) &= \exp\left(\frac{1}{q-1} t(y)\mathfrak{n}_{\sigma,t}\right)\sigma(\Phi) \\
&= \sigma(\Phi)\exp\left(\frac{q}{q-1} t(y)\mathfrak{n}_{\sigma,t}\right) \\
&= \sigma(\Phi' y^{-1})\exp(-t(y^{-1})\mathfrak{n}_{\sigma,t})\exp\left(\frac{1}{q-1} t(y)\mathfrak{n}_{\sigma,t}\right) \\
&= \sigma_{\Phi'}(\Phi) \circ A_{\Phi,\Phi',\sigma,t}
\end{aligned}$$

Finally, it is clear that $A_{\Phi,\Phi',\sigma,t}$ commutes with $\mathfrak{n}_{\sigma,t}$ from expanding the series for exp. Therefore $A_{\Phi,\Phi',\sigma,t}$ is a morphism of Weil-Deligne representations. It is clear that $A_{\Phi',\Phi,\sigma,t} = \exp(\frac{1}{q-1}t(y^{-1})\mathfrak{n}_{\sigma,t})$ its inverse. Moreover, the naturality of $A_{\Phi,\Phi',\sigma,t}$ in σ follows easily from (2.16).

Proof of claim (ii)

For (ii), we need to show that the Weil-Deligne representations $(\rho, V, \mathfrak{n})$ and $(\rho, V, \alpha\mathfrak{n})$ are isomorphic where $t' = \alpha t$, $\rho = \sigma_\Phi$, $\mathfrak{n} = \frac{1}{\alpha}\mathfrak{n}_{\sigma,t}$.
By lemma 2.3.2, $\rho(\Phi)^d$ commutes with $\rho(\mathcal{W}_F)$ for some positive integer d. Consider the "generalized eigenspaces" $V_\lambda := \ker(\rho(\Phi)^d - \lambda\,\mathrm{Id})^{\dim V}$ for $\lambda \in$. Centrality of $\rho(\Phi)^d$ implies that each V_λ is $\rho(\mathcal{W}_F)$-subspace. Moreover, the relation 2.15 implies that $\mathfrak{n}V_{q^d\lambda} \subset V_\lambda$. Since $V = \oplus_\lambda V_\lambda$, pick we can define $B \in \mathrm{End}_K(V)$ by

$$Bv = \mu_\lambda v, \quad v \in V_\lambda$$

where $\{\mu_\lambda\}_\lambda \subset \mathbb{Z}_\ell^\times$ such that $\alpha\mu_{\lambda q^a} = \mu_\lambda$. Then B commutes with $\rho(\mathcal{W}_F)$ on each V_λ since its a scalar, and hence on all of V, and for $v \in V_\lambda$,

$$B\mathfrak{n}v = \mu_{q^{-d}\lambda}\mathfrak{n}v = \alpha\mu_\lambda\mathfrak{n}v = \alpha\mathfrak{n}Bv$$

Therefore, $B\mathfrak{n} = \alpha\mathfrak{n}B$, and hence B is an isomorphism from $(\rho, V, \mathfrak{n})$ and $(\rho, V, \alpha\mathfrak{n})$.

The bijection between isomorphism classes follows easily from (i),(ii) and the observation that the equivalences $D_{\Phi,t}$ preserve dimensions of representations. □

We now fix $K = \overline{\mathbb{Q}}_\ell$, and consider refinements of the equivalence above.
By the remark in chapter 1, it follows that there is an equivalence of categories from $\mathrm{Rep}^f_{\overline{\mathbb{Q}}_\ell}(\mathcal{W}_F)$ to $\mathrm{Rep}^f_{\mathbb{C}}(\mathcal{W}_F)$, and D-$\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathcal{W}_F)$ to D-$\mathrm{Rep}_{\mathbb{C}}(\mathcal{W}_F)$, both dependent only on a choice of isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. In particular we

can transfer all the results from the previous section to finite dimensional smooth representations of $\mathcal{W}_F$ over $\overline{\mathbb{Q}}_\ell$.
We now transfer the notion of semisimplicity of smooth representations from Weil-Deligne representations to ℓ-adic representations via the equivalence above.

Proposition 2.4.5. *Let (σ, V) be an ℓ-adic representation over $\overline{\mathbb{Q}}_\ell$. The following are equivalent:*

(i) For a Frobenius element $\Phi \in \mathcal{W}_F$, the smooth representation (σ_Φ, V) is semisimple.

(ii) For a Frobenius element $\Phi \in \mathcal{W}_F$, $\sigma(\Phi)$ is semisimple.

(iii) $\sigma(g)$ is semisimple for all $g \in \mathcal{W}_F - \mathcal{I}_F$.

Proof. The equivalence of (i) and (ii) follows easily from the fact that $\sigma_\Phi(\Phi) = \sigma(\Phi)$, and proposition 2.3.6. Moreover (iii) clearly implies (ii), so it suffices to show (i) implies (iii)
Let the smooth representation (σ_Φ, V) be semisimple, and $\mathfrak{n} = \mathfrak{n}_{\sigma,t}$. If $g = \Phi^a x \in \mathcal{W}_F - \mathcal{I}_F$, for some nonzero $a \in \mathbb{Z}$ and $x \in \mathcal{I}_F$, then,

$$\begin{aligned}
&\exp\left(\frac{-1}{q^a - 1} t(x)\mathfrak{n}\right) \sigma(g) \exp\left(\frac{1}{q^a - 1} t(x)\mathfrak{n}\right) \\
&= \sigma(g) \exp\left(\frac{-q^a}{q^a - 1} t(x)\mathfrak{n}\right) \exp\left(\frac{1}{q^a - 1} t(x)\mathfrak{n}\right) \\
&= \sigma(g) \exp(-t(x)\mathfrak{n}) = \sigma_\Phi(g)
\end{aligned}$$

using $\|g\| = q^{-a}$ and corollary 2.4.2.1. But $\sigma_\Phi(g)$ is semisimple by proposition 2.3.6, hence so is $\sigma(g)$. ☐

We make a definition:

Definition. A Weil-Deligne representation $(\sigma, V, \mathfrak{n})$ or an ℓ-adic representation (σ, V) of $\mathcal{W}_F$ is said to be ***F*-semisimple** (or Frobenius semisimple), if for any Frobenius element $\Phi \in \mathcal{W}_F$, $\sigma(\Phi)$ is semisimple.

The previous proposition then gives the following:

Theorem 2.4.6. *The equivalence of categories $D_{\Phi,t}$ from theorem 2.4.4 restricts to an equivalence between the full subcategories of F-semisimple representations ℓ-adic and Weil-Deligne representations over $\overline{\mathbb{Q}}_\ell$. Moreover, this equivalence induces a bijection between:*

(i) isomorphism classes of n-dimensional, F-semisimple, ℓ-adic representations over $\overline{\mathbb{Q}}_\ell$

(ii) isomorphism classes of n-dimensional, F-semisimple, Weil-Deligne representations over $\overline{\mathbb{Q}}_\ell$

independent of choices of Φ and t. These are further in bijection with isomorphism classes of n-dimensional, F-semisimple, Weil-Deligne representations over $\mathbb{C}$, with this bijection dependent only on choice of an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$.

Remark. Note that proposition 2.3.6 says that F-semisimplicity for a Weil-Deligne representation $(\sigma, V, \mathfrak{n})$ is equivalent to semisimplicity for the smooth representation (σ, V) but we choose to not call $(\sigma, V, \mathfrak{n})$ semisimple, since a $\mathcal{W}_F$-subreprepresention of V might not be closed under $\mathfrak{n}$. In fact, instead of separately keeping track of a nilpotent endomorphism, one can consider Weil-Deligne representations as representations of a "Weil-Deligne group", where semisimplicity of (σ, V) as a $\mathcal{W}_F$-representation will not equate to semisimplicity of $(\sigma, V, \mathfrak{n})$ as a representation of the Weil-Deligne group. For more details about the Weil-Deligne group, we refer the reader to [Roh94] or Tate's article (Number Theoretic Background) in [BC79].

2.5 Structure of Weil-Deligne Representations

We discuss the operations on Weil-Deligne representations, transferring standard constructions of representations from the category of ℓ-adic representations using the equivalence discussed in the previous section. From now, we will go back to considering smooth and Weil-Deligne representations only over $\mathbb{C}$.

Fix a Frobenius element $\Phi \in \mathcal{W}_F$, and a continuous surjective homommorphism $t : \mathcal{I}_F \to \mathbb{Z}_\ell$.

Dual

Let (σ, V) be an ℓ-adic representation. For $\mathfrak{m} \in \mathrm{End}(V)$, denote its transpose by $\mathfrak{m}^* \in \mathrm{End}(V^*)$, where V^* is the linear dual of V. Then we have a dual ℓ-adic representation (σ^*, V^*) given by $\sigma^*(g) = \sigma(g^{-1})^*$. Then one notes that for x in an open subgroup of $\mathcal{I}_F$,

$$\sigma^*(x) = \sigma(x^{-1})^* = \exp(t(x^{-1})\mathfrak{n}_{\sigma,t})^* = \exp(t(x)(-\mathfrak{n}_{\sigma,t}^*))$$

Therefore, $\mathfrak{n}_{\sigma^*,t} = -\mathfrak{n}^*_{\sigma,t}$. Moreover, on checking separately for powers of Φ and elements of $\mathcal{I}_F$, it follows quickly that $((\sigma^*)_\Phi, V^*)$ is the smooth dual of (σ_Φ, V).
Hence we define the dual of a Weil-Deligne representation $(\rho, W, \mathfrak{n})$ to be:

$$(\rho, W, \mathfrak{n})^\vee = (\check{\rho}, \check{W}, -\check{\mathfrak{n}})$$

where $(\check{\rho}, \check{W})$ is the smooth dual of (ρ, W), and $\check{\mathfrak{n}} = \mathfrak{n}^*$ is the transpose operator. Note that since $\ker \sigma$ is open, $\check{W} = W^*$.

Tensors and Sums

Now let (σ, V) and (τ, W) be ℓ-adic representations. Then we have the tensor product ℓ-adic representation $(\sigma \otimes \tau, V \otimes W)$ given by $\sigma \otimes \tau(g) = \sigma(g) \otimes \tau(g)$. Then for x in an open subgroup of $\mathcal{I}_F$,

$$\begin{aligned}
(\sigma \otimes \tau)(x) &= \sigma(x) \otimes \tau(x) \\
&= \exp(t(x)\mathfrak{n}_{\sigma,t}) \otimes \exp(t(x)\mathfrak{n}_{\tau,t}) \\
&= \sum_{i,j \geq 0} \frac{(t(x))^i \mathfrak{n}^i_{\sigma,t}}{i!} \otimes \frac{(t(x))\mathfrak{n}^j_{\tau,t}}{j!} \\
&= \sum_{i,j \geq 0} \left(\frac{(t(x)(\mathfrak{n}_{\sigma,t} \otimes \mathrm{Id}_W))^i}{i!} \right) \left(\frac{(t(x)(\mathrm{Id}_V \otimes \mathfrak{n}_{\tau,t}))^j}{j!} \right) \\
&= \exp(t(x)(\mathfrak{n}_{\sigma,t} \otimes \mathrm{Id}_W)) \exp(t(x)(\mathrm{Id}_V \otimes \mathfrak{n}_{\tau,t})) \\
&= \exp(t(x)(\mathfrak{n}_{\sigma,t} \otimes \mathrm{Id}_W + \mathrm{Id}_V \otimes \mathfrak{n}_{\tau,t}))
\end{aligned}$$

So we have $\mathfrak{n}_{\sigma\otimes\tau,t} = \mathfrak{n}_{\sigma,t} \otimes \mathrm{Id}_W + \mathrm{Id}_V \otimes \mathfrak{n}_{\tau,t}$. One can similarly check that $((\sigma \otimes \tau)_\Phi, V \otimes W)$ is the tensor product of (σ_Φ, V) and (τ_Φ, W). So naturally, for Weil-Deligne representations $(\rho_i, U_i, \mathfrak{n}_i)$, $i = 1, 2$, we define their tensor product to be,

$$(\rho_1, U_1, \mathfrak{n}_1) \otimes (\rho_2, U_2, \mathfrak{n}_2) = (\rho_1 \otimes \rho_2, U_1 \otimes U_2, \mathfrak{n}_1 \otimes \mathrm{Id}_{U_1} + \mathrm{Id}_{U_2} \otimes \mathfrak{n}_2)$$

We define the direct sum of the $(\rho_i, U_i, \mathfrak{n}_i)$ by,

$$(\rho_1, U_1, \mathfrak{n}_1) \oplus (\rho_2, U_2, \mathfrak{n}_2) = (\rho_1 \oplus \rho_2, U_1 \oplus U_2, \mathfrak{n}_1 \oplus \mathfrak{n}_2)$$

The verification that this definition corresponds to the usual direct sum of ℓ-adic representations is similar (and in fact easier!) to that of duals and tensors as above, so we omit it.

Now that we have these standard constructions, we try to understand what Weil-Deligne representations actually look like. We start with some examples.

(i) If (ρ, V) is any finite dimensional smooth representation of $\mathcal{W}_F$, then $(\rho, V, 0)$ is trivially a Weil-Deligne representation, which we will sometimes denote by just ρ. This is F-semisimple iff (ρ, V) is semisimple (Proposition 2.3.6).

(ii) We define a Weil-Deligne representation on $\mathbb{C}^n$. For $x \in \mathcal{W}_F$, set $\rho(x)e_i = \|x\|^{i-\frac{n-1}{2}} e_i$, $i = 0, \ldots, n-1$, where $\{e_i\}_{i=0}^{n-1}$ is the standard basis of $\mathbb{C}^n$. Further, let $\mathfrak{n} \in \mathrm{M}_n(\mathbb{C})$ given by $\mathfrak{n}e_i = e_{i+1}$, $i = 1, \ldots, n-1$, $\mathfrak{n}e_n = 0$. Then one quickly verifies that $(\rho, \mathbb{C}^n, \mathfrak{n})$ is an F-semisimple Weil-Deligne representation. This is called the **special representation of dimension** n, and is denoted by $\mathrm{sp}(n)$.

(iii) Consider the tensor product $\rho \otimes \mathrm{sp}(n)$ of the previous two examples, in particular when (σ, V) is semisimple. Tensor product of F-semisimple representations is F-semisimple, since the product of eigenbases for a Frobenius element will give an eigenbasis for the Frobenius element of the tensor product representation, so $\rho \otimes \mathrm{sp}(n)$ is F-semisimple.

These essentially cover all the F-semisimple ones. Call a Weil-Deligne representation **indecomposable** if it cannot be obtained as a direct sum of non-trivial Weil-Deligne representations. Note that $\mathrm{sp}(n)$ is indecomposable; if $\mathbb{C}^n = U \oplus V$ for $\mathcal{W}_F$-subspaces U and V closed under $\mathfrak{n}$, then $\ker \mathfrak{n} = (\ker \mathfrak{n} \cap U) \oplus (\ker \mathfrak{n} \cap V)$ which is not possible since $\ker \mathfrak{n}$ has dimension one and both $\ker \mathfrak{n} \cap U$ and $\ker \mathfrak{n} \cap V$ are non-trivial since $\mathfrak{n}$ is nilpotent. Generally, we have:

Theorem 2.5.1. *Every indecomposable, F-semisimple Weil-Deligne representation is of the form $\tau \otimes \mathrm{sp}(n)$ for an irreducible smooth representation τ of $\mathcal{W}_F$ and $n \geq 1$.*

Proof. This proof is borrowed from [Del73].
Let $(\rho, V, \mathfrak{n})$ be an indecomposable F-semisimple Weil-Deligne representation. Consider the decomposition of V into isotypic components for smooth irreducible $\mathcal{W}_{\mathcal{F}}$ representations,

$$V = \bigoplus_{\tau} V^{\tau}$$

where the sum ranges over isomorphism classes of irreducible $\mathcal{W}_F$ representations. The condition 2.15 says exactly that $\mathfrak{n}$ is $\mathcal{W}_F$-map from the smooth representation $(\rho \otimes \|\cdot\|, V)$ to (ρ, V). Since the $(\tau \otimes \|\cdot\|)$-isotypic component in $(\rho \otimes \|\cdot\|, V)$ is V^τ, $\mathfrak{n}V^\tau \subset V^{\tau\otimes\|\cdot\|}$. It follows that, $\oplus_{i\in\mathbb{Z}} V^{\tau\otimes\|\cdot\|^i}$ is closed under both $\mathfrak{n}$ and $\mathcal{W}_F$. If we group together isotypic components which differ by tensoring with an integer power of $\|\cdot\|$, we get a direct sum decomposition of $(\rho, V, \mathfrak{n})$, so from indecomposability it follows that,

$$V = \bigoplus_{i\in\mathbb{Z}} V^{\tau\otimes\|\cdot\|^i}$$

Since V is finite dimensional, only finitely many of these isotypic components are non-trivial. We can replace τ by $\tau \otimes \|\cdot\|^i$ for a suitable i such that $V^{\tau\otimes\|\cdot\|^i} = (0)$ for $i < 0$ and $V^\tau \neq (0)$. So finally we have something of the form,

$$V = \bigoplus_{i=0}^{n-1} V^{\tau\otimes\|\cdot\|^i}$$

Where $n-1$ is the largest i such that $V^{\tau\otimes\|\cdot\|^i} \neq (0)$. Let (τ, W) be a smooth irreducible representation of $\mathcal{W}_F$ of type τ, and set,

$$S_i = \mathrm{Hom}_{\mathcal{W}_F}(\tau \otimes \|\cdot\|^i, V) = \mathrm{Hom}_{\mathcal{W}_F}(\tau \otimes \|\cdot\|^i, V^{\tau\otimes\|\cdot\|^i})$$
$$S = \oplus_{i=0}^{n-1} S_i = \mathrm{Hom}_{\mathcal{W}_F}\left(\bigoplus_{i=0}^{n-1}(\tau \otimes \|\cdot\|^i), V\right)$$

Here we are using τ to denote the representation (τ, W) and not just its isomorphism class. Note that composition by $\mathfrak{n}$ is a nilpotent endomorphism of S. It maps S_i to S_{i+1}; if $f \in S_i = \mathrm{Hom}_{\mathcal{W}_F}(\tau \otimes \|\cdot\|^i, V^{\tau\otimes\|\cdot\|^i})$, it follows from our initial discussion that the image of $\mathfrak{n} \circ f$ lies in $V^{\tau\otimes\|\cdot\|^{i+1}}$, it only remains to show that this is gives a $\mathcal{W}_F$-map. But that follows from,

$$\mathfrak{n} \circ f(\tau(g)\|g\|^{i+1}w) = \mathfrak{n}(\sigma(g)f(\|g\|w)) = \sigma(g)\mathfrak{n} \circ f(w)$$

We construct a Weil-Deligne representation $(\rho', S, \mathfrak{n}')$ as follows:

$$\rho'(g)(f) = \|g\|^i f, \quad f \in S_i$$
$$\mathfrak{n}' f = \mathfrak{n} \circ f, \quad f \in S$$

We then have the following isomorphism of Weil-Deligne representations:

$$F : (\tau, W, 0) \otimes (\rho', S, \mathfrak{n}') \to (\rho, V, \mathfrak{n})$$
$$w \otimes f_i \mapsto f_i(w), \quad w \in W, f_i \in S_i$$

First, to check that this is an isomorphism of smooth $\mathcal{W}_F$ representations, note that the map above is the direct sum of maps:

$$(\tau, W) \otimes (\|\cdot\|^i, S_i) \to (\rho, V^{\tau\otimes\|\cdot\|^i})$$
$$w \otimes f_i \mapsto f_i(w), \quad w \in W, f_i \in S_i$$

for $0 \le i < n$. These can easily be seen to be $\mathcal{W}_F$-isomorphisms by considering a decomposition of $V^{\tau\otimes\|\cdot\|^i}$ into a direct sum of copies of $\tau \otimes \|\cdot\|^i$, and the induced decomposition of $S_i = \mathrm{Hom}_{\mathcal{W}_F}(\tau \otimes \|\cdot\|^i, V^{\tau\otimes\|\cdot\|^i})$ into a direct sum of copies of $\mathrm{End}_{\mathcal{W}_F}(\tau \otimes \|\cdot\|^i) = \mathbb{C}$, and noting that

$$(\tau, W) \otimes (\|\cdot\|^i, \mathbb{C}) \to (\tau \otimes \|\cdot\|^i, W)$$
$$w \otimes c \mapsto cw, \quad w \in W, c \in \mathbb{C}$$

is a $\mathcal{W}_F$-isomorphism. Finally, $F \circ (\mathrm{Id}_W \otimes \mathfrak{n}') = \mathfrak{n} \circ F$ follows from definitions of F and $\mathfrak{n}'$.

We now show that $(\rho', S, \mathfrak{n}')$ is isomorphic to $\|\cdot\|^{\frac{n-1}{2}} \otimes \mathrm{sp}(n)$, with which it will follow that $(\rho, V, \mathfrak{n}) \cong (\|\cdot\|^{\frac{n-1}{2}} \otimes \tau) \otimes \mathrm{sp}(n)$ which is of the required form. The indecomposability of $(\rho, V, \mathfrak{n})$ implies the same for $(\rho', S, \mathfrak{n}')$. Pick a non-zero $v_0 \in S_0$, and let $1 \le k \le n$ be the smallest integer such that $(\mathfrak{n}')^k v_0 = 0$. Set $v_i = (\mathfrak{n}')^i v_0 \in S_i$ for $1 \le i < k$. Pick a linear functional $t : S_{k-1} \to \mathbb{C}$, such that $t(cv_i) = c$. Define $T_i := \ker t \circ (\mathfrak{n}')^{k-i-1} : S_i \to \mathbb{C}$ for $0 \le i < k$, and $T_i = S_i$ for $k \le i < n$. Then $\mathfrak{n}' T_i \subset T_{i+1}$, and for $0 \le i < k$, $S_i = \mathbb{C}v_i \oplus T_i$. But this gives a decomposition of $(\rho', S', \mathfrak{n}')$; the complementary subspaces $\oplus_{i=0}^{k-1} \mathbb{C}v_i$ and $\oplus_{i=0}^{n-1} T_i$ are both closed under $\mathfrak{n}'$ and the action of $\mathcal{W}_F$. Indecomposability lets us conclude that $S = \oplus_{i=0}^{k-1} \mathbb{C}v_i$, in particular $k = n$. Then $e_i \mapsto v_i$ gives an isomorphism from $\|\cdot\|^{\frac{n-1}{2}} \otimes \mathrm{sp}(n)$ to $(\rho', S, \mathfrak{n}')$. □

2.6 L-functions and local constants

We associate L-functions and local constants to F-semisimple Weil-Deligne representations, starting with doing the same for finite dimensional semisimple smooth representations of $\mathcal{W}_F$.

Definition. The **L-function** of a finite dimensional semisimple smooth representation (ρ, V) of $\mathcal{W}_F$ is the function given by

$$L(\rho, s) = \det(1 - \rho_{\mathcal{I}}(\Phi) q^{-s})^{-1}$$

where $\rho_{\mathcal{I}}$ is the subrepresentation of ρ on $V^{\mathcal{I}_F}$ and Φ is a Frobenius element of $\mathcal{W}_F$. Note that $\rho_{\mathcal{I}}$ factors through $\mathcal{W}_F/\mathcal{I}_F$, so the above definition does not depend on the choice of Φ.

Proposition 2.6.1. *The L-function defined above is characterized by the following properties:*

(i) For a character χ of $\mathcal{W}_F$,

$$L(\chi, s) = L(\chi \circ \boldsymbol{a}_F^{-1}, s)$$

where the L-function on the right is the one attached by Tate to the character $\chi \circ \boldsymbol{a}_F^{-1}$ of $F^\times$ (see last section of the first chapter).

(ii) For an irreducible smooth representation (ρ, V) of dimension atleast 2,

$$L(\rho, s) = 1$$

(iii) For finite dimensional semisimple smooth representations $(\rho_i, V_i), i = 1, 2$ of $\mathcal{W}_F$,

$$L(\rho_1 \oplus \rho_2, s) = L(\rho_1, s)L(\rho_2, s)$$

Proof. The first and third properties follows easily from the definitions. For property 2, note that irreducbility of ρ implies that $V_F^{\mathcal{I}} = 0$ or V. In the first case, $L(\rho, s) = 1$ immediately follows. If $V_F^{\mathcal{I}} = V$, then the representation ρ factors through $\mathcal{W}_F/\mathcal{I}_F = \mathbb{Z}$ which is abelian. Irreduciblity of ρ then implies that it must be 1 dimensional, contradicting the hypothesis.

These properties determine the L-function for all finite dimensional semisimple smooth representations since any such representations is a direct sum of irreducibles, so by (iii), it suffices to fix L-functions for irreducible smooth representations. But those are given by the properties (i) and (ii) above, for one dimensional and higher dimensional representations respectively. $\square$

Defining the local constant is much more difficult, and we shall omit proofs of its existence and its properties. One can refer to section 30 of [BH06] for details.

Denote by $\mathcal{G}^{ss}(E)$ the set of isomorphism classes of finite dimensional semisimple smooth representations of $\mathcal{W}_E$, where E/F is a finite separable extension. As for smooth representations of $\mathrm{GL}_2(F)$, we fix a non-trivial character $\psi \in \widehat{F}$. Moreover, we set $\psi_E = \psi \circ \mathrm{tr}_{E/F} \in \widehat{E}$, where $\mathrm{tr}_{E/F}$ is the field trace.

Theorem 2.6.2. *For E/F ranging over finite extensions in $\bar{F}$, there is a unique family of functions,*

$$\begin{aligned} \mathcal{G}^{ss}(E) &\to \mathbb{C}[q^s, q^{-s}]^\times \\ \rho &\mapsto \varepsilon(\rho, s, \psi_E) \end{aligned}$$

satisfying the following properties:

(i) *If χ is a character of $\mathcal{W}_E$, then*

$$\varepsilon(\chi, s, \psi_E) = \varepsilon(\chi \circ \boldsymbol{a}_E^{-1}, s, \psi_E)$$

Again, here the function on the right is the local constant attached to the character $\chi \circ \boldsymbol{a}_E^{-1}$ of $E^\times$.

(ii) *If $\rho_1, \rho_2 \in \mathcal{G}^{ss}(E)$, then*

$$\varepsilon(\rho_1 \oplus \rho_2, s, \psi_E) = \varepsilon(\rho_1, s, \psi_E)\varepsilon(\rho_2, s, \psi_E)$$

(iii) *If $\rho \in \mathcal{G}^{ss}(E)$ is n-dimensional, and E/K is a finite extension contained in $\bar{F}$, then*

$$\frac{\varepsilon(\operatorname{Ind}_{E/K} \rho, s, \psi_K)}{\varepsilon(\rho, s, \psi_E)} = \frac{\varepsilon(\operatorname{Ind}_{E/K} 1_E, s, \psi_K)^n}{\varepsilon(1_E, s, \psi_E)^n}$$

where 1_E denotes the trivial representation of $\mathcal{W}_E$.

Definition. The function $\varepsilon(\rho, s, \psi)$ attached to a semisimple smooth representation (ρ, V) of $\mathcal{W}_F$ given by the previous theorem is called the **Langlands-Deligne local constant** of ρ, relative to the character $\psi \in \widehat{F}$ and the complex variable s.

We list some of its properties:

Proposition 2.6.3. *Let $\psi \in \widehat{F}, \psi \neq 1$ as above, and $\rho \in \mathcal{G}^{ss}(F)$.*

(i) *There is an integer $n(\rho, \psi)$ such that*

$$\varepsilon(\rho, s, \psi) = q^{n(\rho,\psi)(\frac{1}{2}-s)}\varepsilon(\rho, \tfrac{1}{2}, \psi)$$

(ii) *Let $a \in F^\times$. Then:*

$$\begin{aligned} \varepsilon(\rho, s, a\psi) &= \det \rho(a)\|a\|^{\dim(\rho)(s-\frac{1}{2})}\varepsilon(\rho, s, \psi) \\ n(\rho, a\psi) &= n(\rho, \psi) + v_F(a) \dim \rho \end{aligned}$$

In particular, $n(\rho, \psi)$ depends only on ρ and the level of ψ.

(iii) The local constants satisfy the functional equation

$$\varepsilon(\rho, s, \psi)\varepsilon(\check{\rho}, 1-s, \psi) = \det \rho(-1)$$

We now extend these definitions to the F-semisimple Weil-Deligne representation $(\rho, V, \mathfrak{n})$. Let $(\rho_{\mathfrak{n}}, V_{\mathfrak{n}})$ denote the subrepresentation of $\mathcal{W}_F$ on the subspace $V_{\mathfrak{n}} = \ker \mathfrak{n}$. Set

$$L((\rho, V, \mathfrak{n}), s) = L(\rho_{\mathfrak{n}}, s)$$

For the local constant, consider the dual representation $(\check{\rho}, \check{V}, -\check{\mathfrak{n}})$ and set,

$$\varepsilon((\rho, V, \mathfrak{n}), s, \psi) = \varepsilon(\rho, s, \psi)\frac{L(\check{\rho}, 1-s)}{L(\rho, s)}\frac{L(\rho_{\mathfrak{n}}, s)}{L(\check{\rho}_{(-\check{\mathfrak{n}})}, 1-s)}$$

Note that if $\mathfrak{n} = 0$, then the L-function and the local constants are simply the ones attached to the representation (ρ, V), that is $L((\rho, V, \mathfrak{n}), s) = L(\rho, s)$ and $\varepsilon((\rho, V, \mathfrak{n}), s, \psi) = \varepsilon(\rho, s, \psi)$. Moreover, these definitions are still multiplicative, that is, for Weil-Deligne representations $(\rho_i, V_i, \mathfrak{n}_i)$, $i = 1, 2$,

$$L((\rho_1, V_1, \mathfrak{n}_1) \oplus (\rho_2, V_2, \mathfrak{n}_2), s) = L((\rho_1, V_1, \mathfrak{n}_1), s)L((\rho_2, V_2, \mathfrak{n}_2), s)$$
$$\varepsilon((\rho_1, V_1, \mathfrak{n}_1) \oplus (\rho_2, V_2, \mathfrak{n}_2), s, \psi) = \varepsilon((\rho_1, V_1, \mathfrak{n}_1), s, \psi)\varepsilon((\rho_2, V_2, \mathfrak{n}_2), s, \psi)$$

Example. We close the chapter with a computation of the L-function and the local constant for the Weil-Deligne representation $\chi \otimes \mathrm{sp}(2) = (\rho, \mathbb{C}^2, \mathfrak{n})$, where χ is a character of $\mathcal{W}_F$.The smooth representation $(\rho, \mathbb{C}^2)$ decomposes as $\chi\|\cdot\|^{-\frac{1}{2}} \oplus \chi\|\cdot\|^{\frac{1}{2}}$. Moreover, if $\{\check{e}_0, \check{e}_1\}$ is the basis of $(\mathbb{C}^2)^\vee$ dual to the standard basis of $\mathbb{C}^2$, then we have an isomorphism,

$$\chi^{-1} \otimes \mathrm{sp}(2) \to (\chi \otimes \mathrm{sp}(2))^\vee = (\check{\rho}, (\mathbb{C}^2)^\vee, -\check{\mathfrak{n}})$$
$$e_i \mapsto (-1)^i \check{e}_{1-i}$$

It follows that $\check{\rho} \cong \chi^{-1}\|\cdot\|^{-\frac{1}{2}} \oplus \chi^{-1}\|\cdot\|^{\frac{1}{2}}$, $\rho_{\mathfrak{n}} \cong \chi\|\cdot\|^{\frac{1}{2}}$ and $\check{\rho}_{-\check{\mathfrak{n}}} \cong \chi^{-1}\|\cdot\|^{\frac{1}{2}}$. Then we have,

$$L(\chi \otimes \mathrm{sp}(2), s) = L(\rho_{\mathfrak{n}}, s) = L(\chi\|\cdot\|^{\frac{1}{2}}, s) = L(\chi, s + \tfrac{1}{2})$$
$$L(\check{\rho}_{-\check{\mathfrak{n}}}, s) = L(\chi^{-1}\|\cdot\|^{\frac{1}{2}}, s) = L(\chi^{-1}, s + \tfrac{1}{2})$$
$$L(\rho, s) = L(\chi, s - \tfrac{1}{2})L(\chi, s + \tfrac{1}{2})$$
$$L(\check{\rho}, s) = L(\chi^{-1}, s - \tfrac{1}{2})L(\chi^{-1}, s + \tfrac{1}{2})$$

For local constants, fixing a ψ of level one and using proposition 2.6.2 we get:

$$\varepsilon(\rho, s, \psi) = \varepsilon(\chi\|\cdot\|^{-\frac{1}{2}}, s, \psi)\varepsilon(\chi\|\cdot\|^{\frac{1}{2}}, s, \psi)$$

So we obtain,

$$\varepsilon(\chi \otimes \mathrm{sp}(2), s, \psi) = \varepsilon(\rho, s, \psi)\frac{L(\chi^{-1}, \frac{1}{2} - s)L(\chi^{-1}, \frac{3}{2} - s)L(\chi, s + \frac{1}{2})}{L(\chi, s - \frac{1}{2})L(\chi, s + \frac{1}{2})L(\chi^{-1}, \frac{3}{2} - s)}$$

Now if χ is unramified, using proposition 1.3.17, we get:

$$\varepsilon(\rho, s, \psi) = q^{2(s-\frac{1}{2})}\chi(\Phi)^{-2}$$

where Φ is a Frobenius element. and

$$\begin{aligned}\varepsilon(\chi \otimes \mathrm{sp}(2), s, \psi) &= q^{2(s-\frac{1}{2})}\chi(\Phi)^{-2}\frac{1 - \chi(\Phi)q^{\frac{1}{2}-s}}{1 - \chi(\Phi)^{-1}q^{s-\frac{1}{2}}} \\ &= -q^{s-\frac{1}{2}}\chi(\Phi)^{-1} = -\varepsilon(\chi, s, \psi)\end{aligned}$$

But if χ is ramified, all the L-functions above become trivial, so we have

$$\begin{aligned}L(\chi \otimes \mathrm{sp}(2), s) &= 1 \\ \varepsilon(\chi \otimes \mathrm{sp}(2), s, \psi) &= \varepsilon(\chi\|\cdot\|^{-\frac{1}{2}}, s, \psi)\varepsilon(\chi\|\cdot\|^{\frac{1}{2}}, s, \psi)\end{aligned}$$

Chapter 3

The Local Langlands Correspondence and Elliptic Curves

This chapter is divided into two parts. First, we state the Local Langlands Correspondence for GL_2. Then we discuss Elliptic Curves over Local Fields, in particular their Tate modules and reductions. Finally, we look at the smooth representations corresponding to the Tate module, using the tools we developed in chapter 2.

3.1 The Local Langlands Correspondence for GL_2

Now that we've defined and discussed both the sides of the Langlands correspondence, we can finally state it. Let $\mathcal{G}_2(F)$ denote the set of isomorphism classes of 2-dimensional, F-semisimple, complex Weil-Deligne Representations, and $\mathcal{A}_2(F)$ denote the set of isomorphism classes of irreducible smooth complex representations of $\mathrm{GL}_2(F)$. Throughout this section, we will identify characters of $\mathcal{W}_F$ with characters of $F^\times$ via the reciprocity map $\boldsymbol{a}_F$.

Theorem 3.1.1 (Langlands Correspondence)**.** *Let $\psi \in \widehat{F}, \psi \neq 1$. There is a unique map,*

$$\boldsymbol{\pi} : \mathcal{G}_2(F) \to \mathcal{A}_2(F)$$

which satisfies

$$\begin{aligned} L(\chi \cdot \boldsymbol{\pi}(\rho), s) &= L(\chi \otimes \rho, s), \\ \varepsilon(\chi \cdot \boldsymbol{\pi}(\rho), s, \psi) &= \varepsilon(\chi \otimes \rho, s, \psi), \end{aligned} \tag{3.1}$$

for all $\rho \in \mathcal{G}_2(F)$ and all characters χ of $F^\times$.
The map $\boldsymbol{\pi}$ is a bijection, and the equalities (3.1) hold for all $\psi \in \widehat{F}, \psi \neq 1$.

The uniqueness of such a map follows immediately from the Converse Theorem. We draw some conclusions about $\boldsymbol{\pi}$ from its defining properties.

Let $\mathcal{A}_2(F) = \mathcal{A}_2^0(F) \cup \mathcal{A}_2^2(F)$, where $\mathcal{A}_2^0(F)$ and $\mathcal{A}_2^1(F)$ denote the classes of irreducible principal series and cuspidal representations of $\mathrm{GL}_2(F)$ respectively. Then $\mathcal{A}_2^0(F)$ and $\mathcal{A}_2^1(F)$ are closed under twisting by a character of $F^\times$, moreover, for all $\pi \in \mathcal{A}_2^0(F)$, $L(\pi, s) = 1$. Meanwhile for $\pi \in \mathcal{A}_2^1(F)$, by proposition (i) there exists a character χ such that $L(\chi\pi, s)$ is not constant. We have shown:

Proposition 3.1.2. *Let* $\pi \in \mathcal{A}_2(F)$*. Then* $\pi \in \mathcal{A}_2^0(F)$ *iff* $L(\chi \cdot \pi, s) = 1$ *for all characters* χ *of* $F^\times$*.*

We now consider $\mathcal{G}_2(F) = \mathcal{G}_2^0(F) \cup \mathcal{G}_2^1(F)$, where $\mathcal{G}_2^0(F)$ and $\mathcal{G}_2^1(F)$ denote classes of 2-dimensional, F-semisimple, Weil-Deligne representations $(\rho, V, \mathfrak{n})$ where the smooth representation (ρ, V) is irreducible and reducible respectively. These classes are closed under tensoring by a character of $\mathcal{W}_F$. Moreover, for all $(\rho, V, \mathfrak{n}) \in \mathcal{G}_2^0(F)$, $\mathfrak{n} = 0$ since $\ker \mathfrak{n}$ is a $\mathcal{W}_F$-subspace of V. In particular, $L((\rho, V, \mathfrak{n}), s) = L(\rho, s) = 1$ by proposition 2.6.1.

For $(\rho, V, \mathfrak{n}) \in \mathcal{G}_2^1(F)$, by theorem 2.5.1, $(\rho, V, \mathfrak{n}) \cong (\chi_1, \mathbb{C}, 0) \oplus (\chi_2, \mathbb{C}, 0)$ or $(\rho, V, \mathfrak{n}) \cong (\chi_1, \mathbb{C}, 0) \otimes \mathrm{sp}(2)$, for characters χ_1, χ_2 of $\mathcal{W}_F$. In either case, $\chi_1^{-1} \otimes \rho$ has a non-trivial L-function. Again we have,

Proposition 3.1.3. *Let* $\rho \in \mathcal{G}_2(F)$*. Then* $\rho \in \mathcal{G}_2^0(F)$ *iff* $L(\chi \otimes \pi, s) = 1$ *for all characters* χ *of* $\mathcal{W}_F$*.*

We immediately obtain:

Proposition 3.1.4. *Let* $\boldsymbol{\pi}$ *be as in theorem 3.1.1. The* $\boldsymbol{\pi}(\mathcal{G}_2^0(F)) = \mathcal{A}_2^0(F)$ *and* $\boldsymbol{\pi}(\mathcal{G}_2^1(F)) = \mathcal{A}_2^1(F)$*.*

We will only explicitly give the correspondence between $\mathcal{G}_2^1(F)$ and $\mathcal{A}_2^1(F)$. The remaining much harder part of the correspondence is essentially the subject of most of the book [BH06].

Theorem 3.1.5. *There is a unique map*

$$\boldsymbol{\pi}^1 : \mathcal{G}_2^1(F) \to \mathcal{A}_2^1(F)$$

such that

$$L(\chi \cdot \boldsymbol{\pi}^1(\rho), s) = L(\chi \otimes \rho, s)$$

for all $\rho \in \mathcal{G}_2^1(F)$ *and characters* χ *of* $F^\times$*. The map* $\boldsymbol{\pi}^1$ *is bijective, and it satisfies*

$$\boldsymbol{\pi}^1(\chi \otimes \rho) = \chi \cdot \boldsymbol{\pi}^1(\rho),$$
$$\varepsilon(\chi \cdot \boldsymbol{\pi}^1(\rho), s, \psi) = \varepsilon(\chi \otimes \rho, s, \psi),$$

for all $\rho \in \mathcal{G}_2^1(F)$*, characters* χ *of* $F^\times$ *and* $\psi \in \widehat{F}, \psi \neq 1$*.*

Proof. The uniqueness of such a map follows from the Converse theorem for the Principal series. As before, by theorem 2.5.1, any Weil-Deligne representation is of the form $(\rho, V, \mathfrak{n}) \cong (\chi_1, \mathbb{C}, 0) \oplus (\chi_2, \mathbb{C}, 0)$ or $(\rho, V, \mathfrak{n}) \cong (\chi_1, \mathbb{C}, 0) \otimes \mathrm{sp}(2)$, for characters χ_1, χ_2 of $\mathcal{W}_F$. We define the map by considering cases:

(i) $(\rho, V, \mathfrak{n}) \cong (\chi_1, \mathbb{C}, 0) \oplus (\chi_2, \mathbb{C}, 0)$, $\chi_1\chi_2^{-1} \neq \|\cdot\|^{\pm 1}$. Then set $\boldsymbol{\pi}^1(\rho) = \iota_B^G(\chi_1 \otimes \chi_2)$.

(ii) $(\rho, V, \mathfrak{n}) \cong (\chi_1, \mathbb{C}, 0) \oplus (\chi_2, \mathbb{C}, 0)$, $\chi_1\chi_2^{-1} = \|\cdot\|^{\pm 1}$. Then we have $\{\chi_1, \chi_2\} = \{\chi\|\cdot\|^{\frac{1}{2}}, \chi\|\cdot\|^{-\frac{1}{2}}\}$. Set $\boldsymbol{\pi}^1(\rho) = \chi \circ \det$.

(iii) $(\rho, V, \mathfrak{n}) \cong \chi \otimes \mathrm{sp}(2)$. Then set $\boldsymbol{\pi}^1(\rho) = \chi\, \mathrm{St}_G$.

The three cases are closed under tensoring by characters, and infact this map satisfies $\boldsymbol{\pi}^1(\chi \otimes \rho) = \chi\boldsymbol{\pi}^1(\rho)$ for all ρ and χ. It can be easily checked that this map satisfies the other required properties using theorem 1.3.19 and the computation at the end of chapter 2. □

3.2 The Tate module of an Elliptic Curve

We follow the exposition in [Sil09]. Let E be an elliptic curve over an arbitrary field K. Recall that an elliptic curve over K is a smooth curve over K of genus 1 with a chosen K-rational point $O \in E(K)$. Given such a curve one can make it into a commutative algebraic group with O as its identity. In particular, for an algebraic closure $\bar{K}$ of K, $E(\bar{K})$ is an abelian group with identity O, and for any subextension $L \subset \bar{K}$ of K, $E(L)$ is a subgroup.

The structure of the torsion in $E(\bar{K})$ is as follows:

Proposition 3.2.1. *Let E be an elliptic curve over K.*

(i) If $m \geq 1$ is an integer such that $\mathrm{char}(K) \nmid m$, then

$$E(\bar{K})[m] \cong \frac{\mathbb{Z}}{m\mathbb{Z}} \times \frac{\mathbb{Z}}{m\mathbb{Z}}.$$

(ii) If $\mathrm{char}(K) = p \neq 0$, then either,

$$E(\bar{K})[p^e] \cong \frac{\mathbb{Z}}{p^e\mathbb{Z}} \quad \forall e \geq 1 \quad \textit{or} \quad E(\bar{K})[p^e] = \{O\} \quad \forall e \geq 1.$$

Here $E(\bar{K})[m]$ denotes the m-torsion of the abelian group $E(\bar{K})$.

Proof. See [Sil09, Corollary III.6.4] □

Now assume that K is perfect, so $\bar{K}$ is Galois over K. Then we have a continuous action of $\Omega_K = \mathrm{Gal}(\bar{K}/K)$ on the points $E(\bar{K})$ with the discrete topology. This action can be as the elements of Ω_K acting coordinate-wise on $\bar{K}$-points of E after embedding E into a projective space using equations over K. This action is linear, since the group multiplication is defined over K, in particular, for any $m \in \mathbb{Z}$, the torsion subgroups $E(\bar{K})[m]$ is closed under the the action of Ω_K.

Now let ℓ be a prime number. From the proposition 3.2.1 above, we have commutative diagrams,

$$\begin{array}{ccc} E(\bar{K})[\ell^{n+1}] & \xrightarrow{[\ell]} & E(\bar{K})[\ell^n] \\ \downarrow \wr & & \downarrow \wr \\ (\frac{\mathbb{Z}}{\ell^{n+1}\mathbb{Z}})^r & \longrightarrow & (\frac{\mathbb{Z}}{\ell^n\mathbb{Z}})^r \end{array} \tag{3.2}$$

for all $n \geq 1$, for suitable choice of vertical isomorphisms. Here $[\ell]$ is the multiplication by ℓ map on the points of E, the bottom map is the natural quotient map, and $r = 0, 1$ or 2, depending on which of the cases of proposition 3.2.1 we are in.

Definition. The **ℓ-adic Tate module** of an elliptic curve E over K, is the (topological) $\mathbb{Z}_\ell$-module defined by the inverse limit,

$$T_\ell(E) = \varprojlim_n E(\bar{K})[\ell^n]$$

with respect to the multipication by ℓ maps in the top row of (3.2). The **rational ℓ-adic Tate module** $V_\ell(E)$ is the $\mathbb{Q}_\ell$-vector space given by $\mathbb{Q} \otimes T_\ell(E)$.

Note that the multipication by ℓ maps are Ω_K-equivariant. Since the action of Ω_K on $E(\bar{K})[\ell^n]$ is continuous and $\mathbb{Z}/\ell^n\mathbb{Z}$-linear, we have a continuous $\mathbb{Z}_\ell$-linear action of Ω_K on $T_\ell(E)$, and hence a continuous $\mathbb{Q}_\ell$-linear action on $V_\ell(E)$, i.e. $V_\ell(E)$ is an ℓ-adic representation of Ω_K. As a $\mathbb{Z}_\ell$-module, the structure of $T_\ell(E)$ can be derived from proposition 3.2.1:

Proposition 3.2.2. *Let $p = \mathrm{char}(K)$ and E be an elliptic curve over K. Then,* *(i)* $T_\ell(E) \cong \mathbb{Z}_\ell \times \mathbb{Z}_\ell$ *if* $\ell \neq p$.

(ii) $T_p(E) \cong \{0\}$ *or* $\mathbb{Z}_p$ *if* $p \neq 0$

Proof. This follows immediately by taking inverse limit of the diagram (3.2) and noting that proposition 3.2.1 says that $r = 2$ if $\ell \neq p$, and if $\ell = p \neq 0$, then $r = 0$ or 1. □

The Tate module when $\ell \neq \mathrm{char}(K)$ carries a lot of information about the elliptic curve. Let $\phi : E_1 \to E_2$ be an isogeny of elliptic curves over K, that is, a $\bar{K}$-morphism which maps the identity of E_1 to that of E_2. Such a map is automatically a group homomorphism on the points (see [Sil09, Theorem III.4.8]), in particular it induces a $\mathbb{Z}_\ell$-linear map ϕ_ℓ between the Tate modules, that is, we have a group homomorphism,

$$\mathrm{Hom}(E_1, E_2) \to \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell(E_1), T_\ell(E_2)), \quad \phi \mapsto \phi_\ell \tag{3.3}$$

where $\mathrm{Hom}(E_1, E_2)$ is the set of isogenies made into an abelian group by the group multiplication on E_2. Then we have the following result,

Theorem 3.2.3. *Let* E_1*,*E_2 *be elliptic curves over* K *and* $\ell \neq \mathrm{char}(K)$ *be a prime. Then the natural map*

$$\mathrm{Hom}(E_1, E_2) \otimes \mathbb{Z}_\ell \to \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell(E_1), T_\ell(E_2))$$

is injective.

Proof. We'll need and prove only that (3.3) is injective. Injectivity of the map induced on the tensor product with $\mathbb{Z}_\ell$ is much harder, for a proof, see [Sil09, Theorem III.7.4].
Let $\phi : E_1 \to E_2$ be an isogeny such that $\phi_\ell = 0$. But that means ϕ maps $E_1(\bar{K})[\ell^n]$ to the identity O_2 of E_2 for all $n \geq 1$, so the inverse image of O_2 under ϕ is infinite. But the inverse image of O_2 is a closed subset of E_1. Since E_1 is irreducible of dimension one, any proper closed subset must be finite. Therefore ϕ maps all of E_1 to O_2, i.e. $\phi = 0$. □

Now for an elliptic curve E, let $\phi \in \mathrm{End}(E) = \mathrm{Hom}(E, E)$. The Tate module $T_\ell(E)$ is a free $\mathbb{Z}_\ell$-module, so one can compute the determinant and trace of the induced $\mathbb{Z}_\ell$-endomorphism of $T_\ell(E)$. Then we have,

Proposition 3.2.4. *Let* E *be an elliptic curve,* $\phi \in \mathrm{End}(E)$ *and* $\ell \neq \mathrm{char}(K)$ *be a prime. If* ϕ_ℓ *is the induced endomorphism of* $T_\ell(E)$*, then we have,*

$$\det(\phi_\ell) = \deg(\phi), \quad \mathrm{tr}(\phi_\ell) = 1 + \deg(\phi) - \deg(1 - \phi),$$

where $\deg(\phi)$ *is the degree of the map* ϕ*, that is the degree of the function field extension* $\bar{K}(E)/\phi^*(\bar{K}(E))$*. In particular, the* $\det(\phi_\ell)$ *and* $\operatorname{tr}(\phi_\ell)$ *lie in* $\mathbb{Z}$ *and do not depend on* ℓ*.*

Proof. See [Sil09, Proposition III.8.6] □

Corollary 3.2.4.1. *Let* E, ϕ *and* ℓ *be as above. Then* ϕ_ℓ *acts semisimply on* $V_\ell(E) = \mathbb{Q} \otimes T_\ell(E)$*. That is, there exists an eigenbasis of* $V_\ell(E)$ *for* ϕ_ℓ *after extending scalars to an algebraic closure* $\overline{\mathbb{Q}}_\ell$ *of* $\mathbb{Q}_\ell$*.*

Proof. If the characteristic polynomial of $\phi_\ell \in \operatorname{End}_{\mathbb{Q}_\ell}(V_\ell(E))$ has distinct roots, we are done. Suppose not, then the characteristic polynomial must be $(X - \frac{1}{2}\operatorname{tr}(\phi_\ell))^2$. But then theorem 3.2.3 along with Cayley-Hamilton implies that $(2\phi - [\operatorname{tr}(\phi_\ell)])^2 \in \operatorname{End}(E)$ is the zero map, where $[\operatorname{tr}(\phi_\ell)]$ is the multiplication by $\operatorname{tr}(\phi_\ell)$ map on E (this makes sense since by the previous proposition, $\operatorname{tr}(\phi_\ell)) \in \mathbb{Z}$).

But the ring $\operatorname{End}(E)$ has no non-zero nilpotent elements, for if $\psi \in \operatorname{End}(E)$ is a non-zero endomorphism, then the closure of its image is a connected closed subset of E larger than a single point, so it must be all of E. It follows that ψ^n has dense image for any $n \geq 1$ so it cannot be the zero map. So we must have $2\phi - [\operatorname{tr}(\phi_\ell)] = 0$. This implies $\phi_\ell = \frac{\operatorname{tr}(\phi_\ell)}{2}\operatorname{Id}$, which is semisimple. □

We will need this corollary later to show that the ℓ-adic representation on the Tate module of an elliptic curve over a local field is F-semisimple.

3.3 Elliptic Curves over Local Fields

Let E be an elliptic curve over a (non-Archimedean) local field F of characteristic zero. We wish to make sense of a reduction of E modulo the maximal ideal $\mathfrak{p}$ of the valuation ring $\mathfrak{o}_F$, and to this end we will need some generalities on Weierstrass equations.

3.3.1 Weierstrass Equations

We temporarily work over an arbitrary field K. A Weierstrass equation over K is a cubic equation of the form

$$\mathcal{W} = \mathcal{W}(x, y) : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{3.4}$$

where the coefficients a_i lie in K. Homogenizing any such equation by substitutions $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$ defines a curve $E_{\mathcal{W}}$ in $\mathbb{P}^2_K$. Note that the only point of this curve lying at infinity (that is, on the line $Z = 0$) is $[0 : 1 : 0]$, and that this point is nonsingular. Moreover, one can define a group structure on the nonsingular locus $E_{\mathcal{W}}^{ns}$ of $E_{\mathcal{W}}$ by the usual geometric group law on an elliptic curve (see [Sil94, III.2.3, Proposition III.2.5]).

Attached to a Weierstrass equation, [Sil09, III.1] defines many quantities, given by integer polynomials in the coefficients a_i. We'll need the **discriminant** Δ and the invariant denoted by c_4 in [Sil09]. We omit their explicit definitions due to their cumbersome nature. The curve $E_{\mathcal{W}}$ can look like the following based on the value of Δ and c_4:

(i) $\Delta \neq 0$: This happens iff the curve $E_{\mathcal{W}}$ is nonsingular, in particular $E_{\mathcal{W}}$ together with the rational point $[0 : 1 : 0] \in E_{\mathcal{W}}(K)$ forms an elliptic curve.

(ii) $\Delta = 0$: In this case, $E_{\mathcal{W}}$ has a unique singular point $S \neq [0 : 1 : 0]$.

   (a) if $c_4 \neq 0$, $E_{\mathcal{W}}$ has two distinct tangent lines at S, and we call S a **node**. The slopes of the tangent lines lie in an atmost quadratic extension of K. If they lie in K, $E_W^{ns} \cong \mathbb{G}_{m,K} = \mathbb{A}_K^1 - \{0\}$, the multiplicative group over K. That is, there are group isomorphisms $E_{\mathcal{W}}^{ns}(L) \cong L^\times$ for all extensions L/K, functorial in L.

   (b) if $c_4 = 0$, $E_{\mathcal{W}}$ has a unique tangent line at S, and we call S a **cusp**. In this case, $E_{\mathcal{W}}^{ns} \cong \mathbb{G}_{a,K} = \mathbb{A}_K^1$, the additive group over K. That is, there are group isomorphisms $E_{\mathcal{W}}^{ns}(L) \cong L^+$ for all extensions L/K, functorial in L.

For proofs see [Sil09, Propositions III.1.4a, III.3.1c, III.2.5]. Note that the maps given in Proposition III.2.5 are isomorphisms of varieties $E_W^{ns} \cong \mathbb{G}_{m,K}$ and $E_{\mathcal{W}}^{ns} \cong \mathbb{G}_{a,K}$ mentioned in the case (ii) above and not just isomorphisms between their points as indicated in the proposition.

Multiple Weierstrass equations can give rise to isomorphic curves. In the case (i) above, we can characterize when can this happen:

Proposition 3.3.1. *Suppose two Weierstrass equations $\mathcal{W}(x, y)$ and $\mathcal{W}'(x', y')$ over K give isomorphic curves, such that the isomorphism maps the point at infinity to itself. Then the equations are related by a linear change of variables of the form*

$$x = u^2x' + r, \quad y = u^3y' + u^2sx' + t$$

where $u \in K^\times$ and $r, s, t \in K$. Moreover if Δ, c_4 and Δ', c_4' are the discriminant and the quantity "c_4" of $\mathcal{W}$ and $\mathcal{W}'$ respectively, then we have $u^{12}\Delta' = \Delta$ and $u^4 c_4' = c_4$.

Proof. See [Sil09, Proposition III.3.1b] for the proof of the first claim. The change of variable formulae for Δ and c_4 follow from computation (see [Sil09, III.1 Table 3.1] for change of variable formulae for coefficients and quantities attached to Weierstrass equations). □

3.3.2 Reduction of Elliptic Curves

Now suppose we have an elliptic curve E over a local field F and $\mathcal{W}$ is a Weierstrass equation for E, that is $E \cong E_{\mathcal{W}}$. Moreover, one can pick $\mathcal{W}$ to have coefficients in $\mathfrak{o}_F$ by using a change of variables as in proposition 3.3.1 with $v_F(u)$ sufficiently large and negative.

Definition. A Weierstrass equation $\mathcal{W}$ over a local field F is said to be **integral**, if all its coefficients lie in $\mathfrak{o}_F$.

Naively, one could take an integral Weierstrass equation $\mathcal{W}$ for E then reduce it to get a Weierstrass equation $\widetilde{\mathcal{W}}$ over the residue field k. Then one could consider $E_{\widetilde{\mathcal{W}}}$ a reduction of E. However this might not be well defined as illustrated by the following example:

Example. Let p be a prime other than 2 or 3. Consider the Weierstrass equation

$$y^2 = x^3 + p$$

over the field $\mathbb{Q}_p(\sqrt[6]{p})$. This equation has discriminant $-432p^2$ and reduces to $y^2 = x^3$ over the residue field $\mathbb{F}_p$, which is a singular Weierstrass equation. However, the change of variables $x = \sqrt[3]{p}x', y = \sqrt{p}y'$ gives the Weirstrass equation

$$y'^2 = x'^3 + 1$$

which has discriminant $-432 = -16 \times 27$, which is non-zero in the residue field, that is, the reduced Weierstrass equation is nonsingular.

To avoid this, one considers only certain integral Weierstrass equations:

Definition. An integral Weierstrass equation for an elliptic curve E over F said to be **minimal**, if $v_F(\Delta)$ is minimum among all integral Weierstrass equations.

Proposition 3.3.2. *Any Elliptic Curve over F has a minimal Weierstrass equation. If an integral Weierstrass equation $\mathcal{W}(x,y)$ and a minimal Weierstrass equation $\mathcal{W}'(x',y')$ for E are related by a change of variables of the form:*

$$x = u^2x' + r, \quad y = u^3y' + u^2sx' + t$$

then $u, r, s, t \in \mathfrak{o}_F$. Moreover, if both are Weierstrass equations are minimal, then $u \in U_F$.

Proof. The existence of a minimal Weierstrass equation follows from the existence of integral Weierstrass equations and the discreteness of the valuation v_F. Moreover, if we have two Weierstrass equations as in the statement, then by proposition 3.3.1, they are related by a change of coordinates of the form above with $u \in F^\times$ and $r, s, t \in F$. However, if Δ and Δ' are the discriminants of the two equations, then by the same proposition, we know $u^12\Delta' = \Delta$. But $v_F(\Delta) \geq v_F(\Delta')$, therefore $v_F(u) \geq 0$, i.e., $u \in \mathfrak{o}_F$. That $r, s, t \in \mathfrak{o}_F$ follows from similar analysis of the change of variable formulae given in [Sil09, III.1 Table 3.1], for details see [Sil09, Proposition VII.1.3]. Moreover, if both the equations are minimal, then $v_F(\Delta) = v_F(\Delta')$, so $v_F(u) = 0$. □

An immediate consequence of this is that given two minimal Weierstrass equations for E, their reductions modulo $\mathfrak{p}$ are related by a standard change of variables over k, giving us a well-defined notion of reduction of E:

Definition. Let E be an elliptic curve, and $\mathcal{W}$ be a minimal Weierstrass equation for E. Then the curve $\widetilde{E} := E_{\widetilde{\mathcal{W}}}$ over the residue field k defined by the reduction $\widetilde{\mathcal{W}}$ of $\mathcal{W}$ modulo $\mathfrak{p}$ is called the **reduction of E modulo $\mathfrak{p}$**.

The reduction $\widetilde{E}$ can be of three types, based on nature of the reduced Weierstrass equation $\widetilde{\mathcal{W}}$ over k as in (3.3.1):

Definition. We say an elliptic curve E over F has

(i) **good (or stable) reduction**, if $\widetilde{E}$ is nonsingular.

(ii) **multiplicative (or semistable) reduction**, if $\widetilde{E}$ has a node. The reduction is further called **split** or **nonsplit** if the slopes of the tangents at the node lie or do not lie in k respectively.

(iii) **additive (or unstable) reduction**, if $\widetilde{E}$ has a cusp.

In cases ii and iii, one also says E has **bad reduction**. These correspond to the conditions i, ii.a and ii.b of the previous section. One concludes that given a minimal Weierstrass equation $\mathcal{W}$ for E with Δ and c_4 as in the last section, E has good reduction if $v(\Delta) = 0$, multiplicative reduction if $v(\Delta) > 0$ and $v(c_4) = 0$ and additive reduction if $v(\Delta) > 0$ and $v(c_4) > 0$.

One has **reduction map** from $E(F)$ to $\widetilde{E}(k)$. First note that we have a well defined map,

$$\mathbb{P}^2_F(F) \to \mathbb{P}^2_k(k)$$
$$P = [x_1 : x_2 : x_3] \mapsto \tilde{P} = [\tilde{x}_1 : \tilde{x}_2 : \tilde{x}_3]$$

where the representative coordinates $[x_1 : x_2 : x_3]$ are chosen such that $x_i \in \mathfrak{o}_F$ with atleast one in U_F. Then using a minimal Weierstrass equation and its reduction to embed E in $\mathbb{P}^2_F$ and $\widetilde{E}$ in $\mathbb{P}^2_k$, we see that the above map restricts to a map from $E(F)$ to $\widetilde{E}(k)$. Moreover the proposition 3.3.2 implies that this map does not depend on the choice of minimal Weierstrass equation.

As noted in the last section, the nonsingular part $\widetilde{E}^{ns}$ has a group structure. Define:

$$E_0(F) = \{P \in E(F) \mid \tilde{P} \in \widetilde{E}^{ns}(k)\}$$

Proposition 3.3.3. *The reduction map $E_0(F) \to \widetilde{E}^{ns}(k)$ is a surjective group homomorphism.*

Proof. The surjectivity follows by Hensels' lemma. The fact that the reduction is a group homomorphism essentially follows from the fact that the group law of an elliptic curve is determined by the property that three points on the curve sum to zero iff they are collinear (under an embedding into $\mathbb{P}^2$ via a Weierstrass equation), and the reduction map sends lines to lines. For details, see [Sil09, Proposition VII.2.1]. □

So we can get a glimpse into the group of points of E via the reduction map. Note that if E has good reduction, then $E_0(F) = E(F)$ and $\widetilde{E}^{ns} = \widetilde{E}$. Let p denote the residue characteristic of F. Then the reduction map preserves the torsion away from p:

Proposition 3.3.4. *Let E be an elliptic curve over F with good reduction, and $m \geq 1$ is an integer coprime to* $\mathrm{char}(k)$, *then the reduction map,*

$$E(F)[m] \to \widetilde{E}(k)$$

is injective. Here, $E(F)[m]$ denotes the m-torsion in the abelian group $E(F)$.

Proof. See [Sil09, Proposition VII.3.1b]. □

Taking a direct limit of the reduction maps over all finite extensions of F inside a fixed algebraic closure $\bar{F}$, one obtains a reduction map $E(\bar{F}) \to \widetilde{E}(\bar{k})$, where we identify an algebraic closure $\bar{k}$ of k with the residue field of $\bar{F}$. Recall that we have actions of Galois groups $\Omega_F = \mathrm{Gal}(\bar{F}/F)$ and $\mathrm{Gal}(\bar{k}/k)$ on $E(\bar{F})$ and $\widetilde{E}(\bar{k})$ respectively. Further recall from chapter 2,that we have an exact sequence,

$$1 \longrightarrow \mathcal{I}_F \longrightarrow \Omega_F \xrightarrow{\sigma \mapsto \tilde{\sigma}} \mathrm{Gal}(\bar{k}/k) \longrightarrow 1,$$

where $\tilde{\sigma} \in \mathrm{Gal}(\bar{k}/k)$ denotes the automorphism of the residue field induced by σ and $\mathcal{I}_F$ is the inertia group of the local field F. Using this we get an action of Ω_F on $\widetilde{E}(\bar{k})$, via the map $\sigma \mapsto \tilde{\sigma}$. Then the reduction map is Ω_F-equivariant:

Lemma 3.3.5. *Let $P \in E(\bar{F})$ and $\sigma \in \Omega_F$. Then we have,*

$$\widetilde{P^\sigma} = \tilde{P}^{\tilde{\sigma}}$$

Proof. Suppose we have embeddings $E \hookrightarrow \mathbb{P}^2_F$ and $\widetilde{E} \hookrightarrow \mathbb{P}^2_k$ given by a minimal Weierstrass equation for E. Then if $P = [x : y : z]$ with $x, y, z \in \mathfrak{o}_F$ with atleast one in U_F,

$$\widetilde{P^\sigma} = [\widetilde{\sigma(x)} : \widetilde{\sigma(y)} : \widetilde{\sigma(z)}] = [\tilde{\sigma}(\tilde{x}) : \tilde{\sigma}(\tilde{y}) : \tilde{\sigma}(\tilde{z})] = \tilde{P}^{\tilde{\sigma}}$$

□

Definition. We call a set S with an action of Ω_F **unramified**, if $\mathcal{I}_F$ acts trivially on S.

Proposition 3.3.6. *Let E be an elliptic curve over F with good reduction. Then:*

(i) Let $m \geq 1$ be an integer coprime to p. The reduction map induces an Ω_F-equivariant isomorphism $E(\bar{K})[m] \cong \widetilde{E}(\bar{k})[m]$.

(ii) Let $\ell \neq p$ be a prime. The reduction map induces an Ω_F-equivariant isomorphism $T_\ell(E) \cong T_\ell(\widetilde{E})$.

In particular, $E(\bar{K})[m]$ and $T_\ell(E)$ are unramified for m and ℓ as above.

Proof. For (i), let $F' \subset \bar{F}$ be the finite extension of F generated by coordinates of all points in $E(\bar{F})[m]$. Since E has good reduction, for a minimal Weierstrass equation for E over F, $v_{F'}(\Delta) = 0 = v_F(\Delta)$, so it must also be minimal over F'. It then follows from proposition 3.3.4 that the reduction map

$$E(F')[m] = E(\bar{F})[m] \to \widetilde{E}(\bar{k})[m]$$

is injective. But for $\sigma \in \mathcal{I}_F$, $\tilde{\sigma}$ is the trivial automorphism, so the result follows from the lemma. Moreover, by proposition 3.2.1, both the domain and codomain have m^2 elements, so any injective map is an isomorphism. The statment (ii) follows from (i) for $m = \ell^n, n \geq 1$. □

Therefore the ℓ-adic representation of Ω_F on $V_\ell(E)$ is unramified, if E has good reduction. Turns out the converse is also true:

Theorem 3.3.7 (Criterion of Néron-Ogg-Shafarevich)**.** *Let E be an elliptic curve over a local field F. Then the following are equivalent:*

(i) E has good reduction.

(ii) $E[m]$ is unramified for all integers $m \geq 1$ coprime to the char(k)*.*

(iii) The Tate module $T_\ell(E)$ is unramified for a prime (all primes) $\ell \neq$ char(k).

(iv) $E[m]$ is unramified for infinitely many integers $m \geq 1$ coprime to char(k).

For a proof, see [Sil09, Theorem VII.7.1].

Corollary 3.3.7.1. *Let $\sigma_\ell : \Omega_F \to \mathrm{Aut}_{\mathbb{Z}_\ell}(T_\ell(E))$ denote the action of Ω_F on the Tate module $T_\ell(E)$ for some $\ell \neq \mathrm{char}(k)$. Then E has good reduction over a finite extension F' of F, iff $\mathcal{I}_{F'} \subset \ker \rho_\ell$.*

Proof. The ℓ-adic representation on the Tate module of E over F' is exactly $\sigma_\ell |_{\Omega_{F}}$. The corollary then follows immediately from the criterion of Néron-Ogg-Shafarevich. □

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We now justify the terms stable, semistable and unstable for good, multiplicative and additive reductions respectively:

Theorem 3.3.8 (Semistable reduction theorem)**.** *Let E be an elliptic curve over F.*

(i) If E has good or multiplicative reduction over F, for any finite extension F'/F, the reduction type of E over F' is the same as that of F.

(ii) There exists a finite extension F'/F such that E has good or (split) multiplicative reduction over F'.

Proof. To see (i), let $\mathcal{W}(x, y)$ and $\mathcal{W}'(x', y')$ be minimal Weierstrass equations over F and F' respectively. Then by proposition 3.3.2, they are related by a change of coordinates,

$$x = u^2x' + r \quad y = u^3y' + su^2x + t$$

where $u, r, s, t \in \mathfrak{o}_{F'}$. Then from proposition 3.3.1, we have $u^{12}\Delta' = \Delta$ and $u^4c_4' = c_4$, where $\Delta, \Delta', c_4, c_4'$ are as in the proposition. But integrality of $\mathcal{W}$ implies $v_{F'}(\Delta), v_{F'}(c_4) \geq 0$. Therefore,

$$0 \leq v_{F'}(u) \leq \min\left\{\frac{1}{12}v_{F'}(\Delta), \frac{1}{4}v_{F'}(c_4)\right\}$$

But then in case of good or multiplicative reduction over F, $v_F(\Delta) = 0$ or $v_F(c_4) = 0$ respectively. In either case, we get $v_{F'}(u) = 0$, so $\mathcal{W}$ is also minimal over F'; in particular, the reduction type over F' is same as that of F.
For (ii), one writes a Weierstrass equation for E in Legendre of Deuring normal form over a finite extension of F, and shows by hand that such an equation must have good or multiplicative reduction, after possibly another change of variables. For details see [Sil09, Proposition VII.5.4]. □

Hence good and multiplicative reduction are "stable" under field extensions, while additive reduction turns into good or multiplicative reduction over some finite extension.

Definition. Let E be an elliptic curve over a local field F. Then we say E has **potentially good (resp. multiplicative) reduction** if there is a finite extension F'/F such that E has good (resp. multiplicative) reduction over F'.

Then the previous theorem says exactly that every elliptic curve E over a local field has either potentially good or potentially multiplicative reduction.

Example. We go back to the example in the beginning of the section. The Weierstrass equation
$$y^2 = x^3 + p$$
has discriminant $\Delta = -432p^2$. In particular, $v_{\mathbb{Q}_p}(\Delta) = 2$. By proposition 3.3.1, a change of variables can only change $v_{\mathbb{Q}_p}(\Delta)$ by 12, therefore this equation is minimal. The reduced equation, $y^2 = x^3$ has just one tangent at $[0:0:1]$, hence the elliptic curve defined by this equation has additive reduction. However as noted earlier, over $\mathbb{Q}_p(\sqrt[6]{p})$, this curve is isomorphic to the one defined by
$$y^2 = x^3 + 1$$
which has good reduction. Therefore, the elliptic curve $y^2 = x^3 + p$ has additive but potentially good reduction over $\mathbb{Q}_p$.

3.3.3 The Smooth Representation associated to an Elliptic Curve

Given an Elliptic Curve over a non-Archimedean local field F of characteristic zero and a prime $\ell \neq p$, the residue characteristic, we have an associated ℓ-adic representation of the Galois group Ω_F on the Tate module $V_\ell(E)$, which we denote by $(\sigma_\ell, V_\ell(E))$. Further denote by $(\sigma_\ell^\vee, \bar{V}_\ell(E)^\vee)$, the dual of the extension of scalars of σ_ℓ to a fixed algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$.
We construct a Weil-Deligne representation out of this as follows: Restrict it to the Weil group $\mathcal{W}_F$, change scalars to $\mathbb{C}$ by choosing an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$, and finally using the construction in theorem 2.4.4. Denote the resulting Weil-Deligne representation by $(\sigma_{\ell,\iota}, V_{\ell,\iota}, \mathfrak{n}_{\ell,\iota})$. We suppress the choices Φ and t required to apply theorem 2.4.4 since we only care about the isomorphism class of the representation.

Theorem 3.3.9. *The Weil-Deligne representation $(\sigma_{\ell,\iota}, V_{\ell,\iota}, \mathfrak{n}_{\ell,\iota})$ is F-semisimple. Moreover its isomorphism class does not depend on choice of ℓ or ι.*

More precisely, this is what happens in the case of potentially good reduction:

Theorem 3.3.10. *Let E be an elliptic curve over a local field F with potentially good reduction. Then $(\sigma_{\ell,\iota}, V_{\ell,\iota}, \mathfrak{n}_{\ell,\iota}) = (\sigma_\ell^\vee|_{\mathcal{W}_F}, \mathbb{C} \otimes_\iota \bar{V}_\ell(E)^\vee, 0)$ is F-semisimple, and the isomorphism class of this representation does not depend on ℓ or ι. Moreover, E has good reduction over F iff σ_ℓ is unramified.*

Proof. Suppose K/F is a finite extension such that E has good reduction over E. The restriction of σ_ℓ to $\mathcal{W}_K$ is the ℓ-adic representation on the Tate module of E as an elliptic curve over K, so by proposition 3.3.6, ρ_ℓ is trivial on $\mathcal{I}_K$, which is an open subgroup of $\mathcal{I}_F$. It follows that $\mathfrak{n}_{\ell,\iota} = 0$ (see theorem 2.4.2), and we have $(\sigma_{\ell,\iota}, V_{\ell,\iota}, \mathfrak{n}_{\ell,\iota}) = (\sigma_\ell^\vee |_{\mathcal{W}_F}, \mathbb{C} \otimes_\iota \bar{V}_\ell(E)^\vee, 0)$. By proposition 2.3.6, F-semisimplicity of this representation is equivalent to semisimplicity of $\sigma_\ell^\vee$.

Since K/F is a finite extension, to show $\sigma_\ell^\vee$ is semisimple, it suffices to show that $\sigma_\ell^\vee |_{\mathcal{W}_K}$ is semisimple, by lemma 2.3.7. Therefore, we reduce to the case E has good reduction over $K = F$. Let $\Phi \in \mathcal{W}_F$ be a Frobenius element. By proposition 2.3.6, we only need to show that $\sigma_{ell}^\vee(\Phi)$ acts semisimply on $\mathbb{C} \otimes_\iota \bar{V}_\ell(E)^\vee$. However, by proposition 3.3.6, this is equivalent to showing that $\tilde{\Phi} = \varphi^{-1} \in \mathrm{Gal}(\bar{k}/k)$ acts semisimply on $\mathbb{C} \otimes_\iota \bar{V}_\ell(\widetilde{E})^\vee$, where $\varphi : x \mapsto x^q$ is the Frobenius automorphism of the finite residue field $k = \mathbb{F}_q$ of F. But the map φ is "algebraic", that is, there is an element $\phi \in \mathrm{End}(\widetilde{E})$, given on coordinates by $\phi : [x : y : z] \mapsto [x^q : y^q : z^q]$, such that $\varphi = \phi_\ell$, the Tate module endomorphism induced by ϕ. But by corollary 3.2.4.1, ϕ_ℓ acts semisimply on $V_\ell(\widetilde{E})$, and hence on its dual.

The independence of isomorphism class from choices of ℓ and ι follows from the following:

Theorem 3.3.11 ([ST68, Corollary to Theorem 3])**.** *Let E be an elliptic curve over a local field F with potentially good reduction and $\ell \neq p$. Further, let $(\sigma_\ell, T_\ell(E))$ be the ℓ-adic representation of Ω_F on the Tate module. Then for $x \in \mathcal{W}_F$, the characteristic polynomial of $\rho_\ell(x)$ has coefficients in $\mathbb{Q}$ and is independent of ℓ.*

Theorem 3.3.12 ([Eti+, Theorem 3.6.2])**.** *Let A be an algebra over $\mathbb{C}$. Characters of all irreducible finite dimensional representations of A are linearly independent.*

By theorem 3.3.11, it follows that for $x \in \mathcal{W}_F$, $\mathrm{tr}(\sigma_\ell(x)) \in \mathbb{Q}$ and is independent of ℓ. Rationality further makes $\mathrm{tr}(\sigma_{\ell,\iota}(x)) = \iota(\mathrm{tr}(\sigma_\ell(x)))$ independent of choice of ι. Therefore the character of the semisimple representation $\mathbb{C}[\mathcal{W}_F]$-representation $\sigma_{\ell,\iota}$ does not depend on ℓ or ι. The result then follows from theorem 3.3.12. Finally, the claim about good reduction iff the smooth representation is unramified is exactly the criterion of Néron-Ogg-Shafarevich. □

We state without proof what happens in the case of potentially multiplicative reduction:

Theorem 3.3.13. *Let E be an elliptic curve over a local field R with potentially multiplicative reduction. There exists a character χ of $\mathcal{W}_F$ such that $\chi^2 = 1$, $(\sigma_E, V_E, \mathfrak{n}_E) \cong \chi\|\cdot\|^{-\frac{1}{2}} \otimes \mathrm{sp}(2)$, and χ is trivial, unramified but non-trivial or ramified if E has split multiplicative, nonsplit multiplicative or additive reduction over F respectively.*

For a proof, see [Roh94, Section 15]. Note that their definition of $\mathrm{sp}(n)$ is different from ours, leading to slightly different statements.
Therefore, we can just write $(\sigma_E, V_E, \mathfrak{n}_E)$ for the above Weil-Deligne representation constructed from E. Since its 2-dimensional and F-semisimple, we can use the Langlands correspondence we stated at the start of this chapter to obtain an irreducible smooth representation $\pi_E := \boldsymbol{\pi}((\sigma_E, V_E, \mathfrak{n}_E))$ of $\mathrm{GL}_2(F)$. Call this **the smooth representation of $\mathrm{GL}_2(F)$ associated to E**. Our goal is to analyze what this looks like based on properties of E. First, a we need a result to handle the case of good reduction.

Theorem 3.3.14 ([Sil09, Proposition II.2.11, Theorem V.2.3.1])**.** *Let E be an elliptic curve over a finite field $\mathbb{F}_q$, and let $\phi \in \mathrm{End}(E)$ be the Frobenius endomorphism. Then,*

$$\deg(\phi) = q, \quad \deg(1 - \phi) = |E(\mathbb{F}_q)|$$

Let $P(T) = T^2 - aT + q$ be the characteristic polynomial of ϕ_ℓ for any $\ell \neq \mathrm{char}(\mathbb{F}_q)$ where $a = (q + 1 - |E(\mathbb{F}_q)|)$ (see proposition 3.2.4), and $\alpha_1, \alpha_2 \in \mathbb{C}$ be its roots. Then $|\alpha_i| = \sqrt{q}$.

Theorem 3.3.15. *Let E be an elliptic curve over a non-Archimedean local field F of characteristic zero, and π_E be the associated irreducible smooth representation of $\mathrm{GL}_2(F)$, as defined above. Then we have the following:*

(i) Suppose E has good reduction over an abelian extension of F, then $\pi_E \cong \iota_E^B(\chi_1 \otimes \chi_2)$ for characters χ_1, χ_2 of $F^\times$ such that $\chi_1\chi_2^{-1} \neq \|\cdot\|^{\pm}$ or $\pi_E \cong \chi \circ \det$ for some character χ of $F^\times$.

(ii) Suppose E has potentially good reduction, but not over an abelian extension of F. Then π_E is a cuspidal representation.

(iii) Suppose E has good reduction over F, and α_1, α_2 are the roots of the polynomial $T^2 - aT + q$, where $a = q + 1 - |\widetilde{E}(\mathbb{F}_q)|$ and q is the size of the residue field. Then $\pi_E \cong \iota_B^G(\chi_1 \otimes \chi_2)$, where χ_i are unramified characters of $F^\times$ determined by $\chi_i(\varpi) = \alpha_i$.

(iv) Suppose E has potentially multiplicative reduction, then $\pi_E \cong \chi\,\mathrm{St}_G$ for a character χ of $F^\times$.

(a) If E has split multiplicative reduction over F, $\chi = \|\cdot\|^{-\frac{1}{2}}$.

(b) If E has nonsplit multiplicative reduction over F, $\chi \neq \|\cdot\|^{-\frac{1}{2}}$ is unramified.

(c) If E has additive reduction over F, χ is ramified.

Proof. Throughout the proof, ℓ denotes a prime number not equal to the residue characteristic p.
(i),(ii) If E has good reduction over an abelian extension K/F, then by corollary 3.3.7.1, the commutator $\mathcal{W}_F^c \subset \mathcal{W}_K \cap \mathcal{I}_F = \mathcal{I}_K \subset \ker \sigma_\ell$. Conversely, If E has good reduction over some extension K/F and $\mathcal{W}_F^c \subset \ker \sigma_\ell$, then if $K' = K \cap F^{ab}$ is the maximal abelian subextension of K, then $\mathcal{W}_{K'} = \mathcal{W}_K \mathcal{W}_F^c$. In particular $\mathcal{I}_{K'} = \mathcal{W}_{K'} \cap \mathcal{I}_F = \mathcal{I}_K \mathcal{W}_F^c \subset \ker \sigma_\ell$, by theorem 3.3.6. Therefore by the criterion of Néron-Ogg-Shafarevich, E has good reduction over the abelian extension K'.

In particular if E has potentially good reduction, it attains good reduction over an abelian extension of F iff $\mathcal{W}_F^c \subset \ker \sigma_\ell$ or equivalently, the σ_ℓ (or equivalently $\sigma_\ell^\vee$) factors through an abelian group. By theorem 3.3.10, $(\sigma_E, V_E, \mathfrak{n}_E) \cong (\sigma_\ell^\vee, \mathbb{C} \otimes_\iota \bar{V}_\ell(E)^\vee, 0)$, and is F-semisimple. If E does not attain good reduction over an abelian extension, the image of $\sigma_\ell^\vee = \sigma_E$ must be non-abelian. This is only possible if $(\sigma_E, V_E, \mathfrak{n}_E) \in \mathcal{G}_2^0(E)$, therefore π_E is a cuspidal representation by proposition 3.1.4.
If E does attain good reduction over an abelian extension, $\sigma_\ell^\vee$ factors through an abelian group, so the semisimple (Proposition 2.3.6) smooth representation $(\sigma_\ell^\vee, \mathbb{C} \otimes_\iota \bar{V}_\ell(E)^\vee)$, decomposes into a direct sum of characters (Corollary 1.2.6.2). Therefore, $(\sigma_E, V_E, \mathfrak{n}_E) \cong (\chi_1, \mathbb{C}, 0) \oplus (\chi_2, \mathbb{C}, 0)$, and (i) follows from the explicit description of the Langlands correspondence in theorem 3.1.5.

(iii) If E has good reduction over F, then we have a Ω_F-equivariant isomorphism $T_\ell(E) \cong T_\ell(\widetilde{E})$. So by theorem 3.3.10, we get $(\sigma_E, V_E, \mathfrak{n}_E) \cong (\tilde{\sigma}_\ell^\vee, \mathbb{C} \otimes_\iota \bar{V}_\ell(\widetilde{E})^\vee, 0)$, where $(\tilde{\sigma}_\ell, \bar{V}_\ell(\widetilde{E}))$ is the extension of scalars to $\overline{\mathbb{Q}}_\ell$ of the ℓ-adic representation of $\mathrm{Gal}(\bar{k}/k)$ on the Tate module of $\widetilde{E}$, seen as a representation of $\mathcal{W}_F$ via the usual map $\mathcal{W}_F \subset \Omega_F \to \mathrm{Gal}(\bar{k}/k)$. But then $\tilde{\sigma}_\ell$ factors through $\mathcal{W}_F/\mathcal{I}_F \cong \mathbb{Z}$. Moreover, $\Phi \in \mathcal{W}_F$ acts as ϕ_ℓ^{-1} on $V_\ell(\widetilde{E})$, where $\phi \in \mathrm{End}(\widetilde{E})$ is the Frobenius endomorphism as in the proof of theorem 3.3.10, and ϕ_ℓ is its induced action on the Tate module. By theorem 3.2.4 and its corollary, we have that ϕ_ℓ acts semisimply with eigenvalues given by roots of the polynomial,

$$T^2 - (1 + \deg(\phi) - \deg(1 - \phi))T + \deg(\phi) = 0$$

which are exactly α_1 and α_2 by theorem 3.3.14. Therefore Φ acts semisimply with eigenvalues α_1^{-1} and α_2^{-1} on $\bar{V}_\ell(\widetilde{E})$). So now we have $(\sigma_E, V_E, \mathfrak{n}_E) \cong (\tilde{\sigma}_\ell^\vee, \mathbb{C} \otimes_\iota \bar{V}_\ell(\widetilde{E})^\vee, 0)$, with $\sigma_\ell^\vee$ factoring through $\mathcal{W}_F/\mathcal{I}_F$ and Φ acting semisimply with eigenvalues α_1 and α_2. It follows that $(\rho_E, V_E, \mathfrak{n}_E) \cong (\chi_1, \mathbb{C}, 0) \oplus (\chi_2, \mathbb{C}, 0)$, with χ_i as in the statement. The claim then follows from theorem 3.1.5, and the fact that $|\alpha_i| = \sqrt{q}$ (theorem 3.3.14), so $\chi_1\chi_2^{-1} \neq \|\cdot\|^{\pm}$.

(iv) This follows immediately from theorems 3.3.13 and 3.1.5. □

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